The mathematical procedure for integrating a step function multiplied by a function $f(x)$ is

$$\int_0^x H_a f(x)\,dx = H_a \int_a^x f(x)\,dx$$

Example 1. Let $f(x) = x^2$.

$$\int_0^x H_a x^2\,dx = H_a \int_a^x x^2\,dx = \left[H_a \frac{x^3}{3}\right]_a^x + C = H_a \frac{x^3 - a^3}{3} + C$$

where C = constant of integration.

Example 2. When $b > a$,

$$\int_0^x H_b H_a (x-a)\,dx = H_b \int_b^x (x-a)\,dx$$

$$= H_b \left[\frac{(x-a)^2}{2}\right]_b^x + C = H_b \frac{(x-a)^2 - (b-a)^2}{2} + C$$

Example 3. The method of handling changes of section by means of step functions is demonstrated as follows: Fig. 5-4 shows a beam having three sections of different moments of inertia. F_1 and F_2 are overhung loads and there are fixed bearings at R_L and R_R. The moment equation valid for any section is

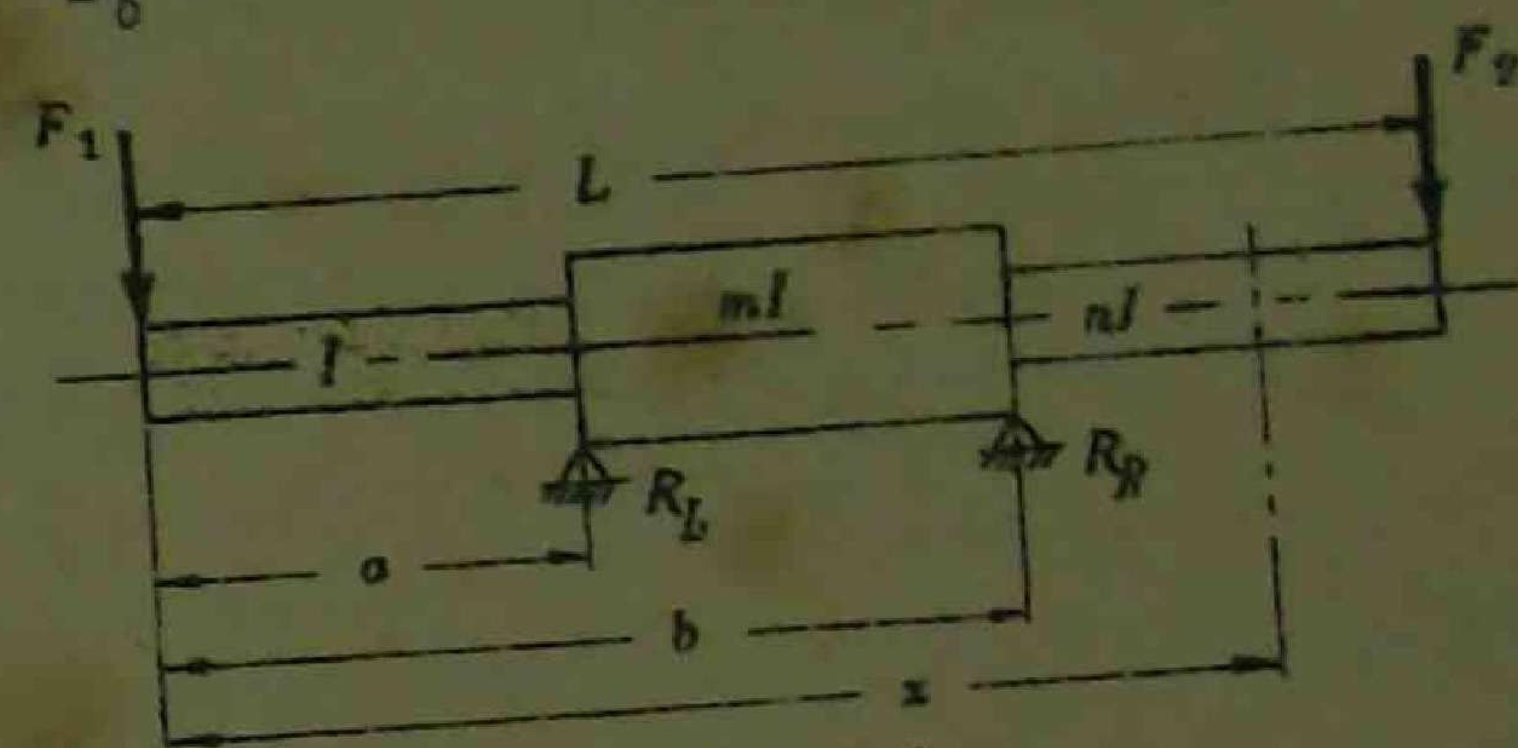


Fig. 5-4

$$M = -F_1 x + R_L (x-a) H_a + R_R (x-b) H_b$$

Now $1/I_x$, the reciprocal of the moment of inertia at any section, may be written

$$\frac{1}{I_x} = \frac{1}{I}\left[1 - H_a + \frac{H_a}{m} - \frac{H_b}{m} + \frac{H_b}{n}\right]$$

from which

$$IE\frac{d^2y}{dx^2} = \left[-F_1 x + R_L (x-a) H_a + R_R (x-b) H_b\right]\left[1 + H_a\left(\frac{1}{m} - 1\right) + H_b\left(-\frac{1}{m} + \frac{1}{n}\right)\right]$$

$$= -F_1 x + R_L (x-a) H_a + R_R (x-b) H_b - F_1 x H_a \left(\frac{1}{m} - 1\right)$$

$$+ R_L (x-a) H_a H_a \left(\frac{1}{m} - 1\right) + R_R (x-b) H_b H_a \left(\frac{1}{m} - 1\right)$$

$$-F_1 x H_b \left(-\frac{1}{m} + \frac{1}{n}\right) + R_L (x-a) H_a H_b \left(-\frac{1}{m} + \frac{1}{n}\right)$$

$$+ R_R (x-b) H_b H_b \left(-\frac{1}{m} + \frac{1}{n}\right)$$

Double integration may be completed as explained above, noting that $H_a H_a = H_a$ and $H_a H_b = H_b$.

DEFLECTION DUE TO SHEAR may be significant, for example, in machine members which are short in comparison to their depth, or for large diameter hollow members. In such cases the deflection due to shear should be added to the deflection due to bending. This could

The drawings and detail specifications for a completed design are a record of a multitude of decisions, some large and some small. The designer, in the later stages of the design process, is basically a decision maker. He must work from a sound basis of scientific principles supplemented by empirical data. However, it must be understood that science can only establish limits within which a decision must be made, or give a statistical picture of the effects of a particular decision. The decision itself is made by the designer. Hence judgment in making decisions is one of the outstanding characteristics of a good designer.

THE DESIGN OF A MACHINE must follow a plan somewhat as shown in the adjacent figure.

After the general specifications have been set, the kinematic arrangement, or skeleton, of the machine must be established. This is followed by a force analysis (incomplete because masses of moving parts are not yet known in designs where dynamics is of importance). With this information the components can be designed (tentatively, because forces are not known exactly). Then a more exact force analysis can be made and the design re-

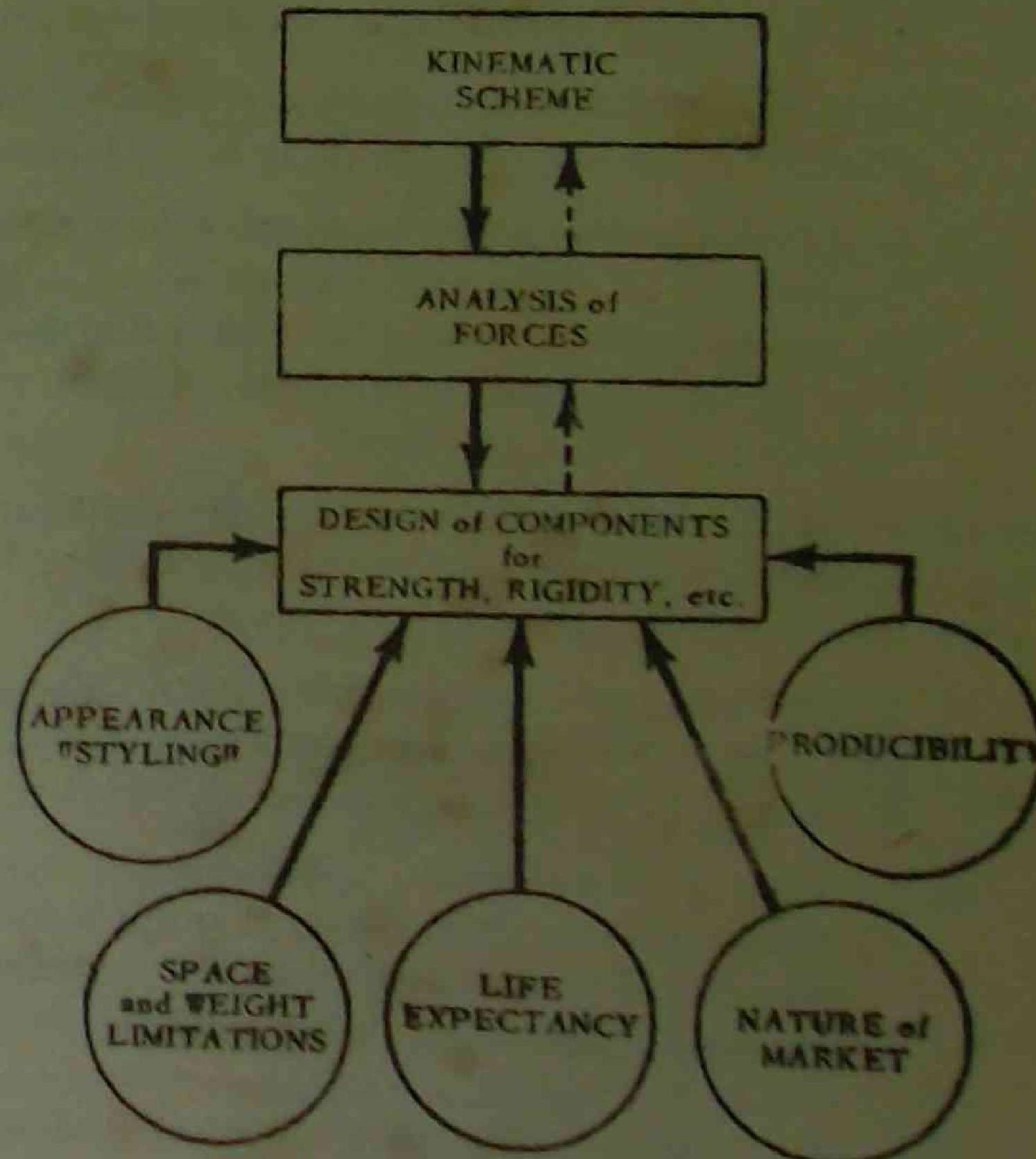


Fig. 1-1

THE MAXIMUM AND MINIMUM NORMAL STRESSES, $s_n(max)$ or $s_n(min)$, which are tensile or compressive stresses, can be determined for the general case of two-dimensional loading on a particle by

$$(1)\quad s_n(max) = \frac{s_x + s_y}{2} + \sqrt{\left(\frac{s_x - s_y}{2}\right)^2 + \tau_{xy}^2}$$

$$(2)\quad s_n(min) = \frac{s_x + s_y}{2} - \sqrt{\left(\frac{s_x - s_y}{2}\right)^2 + \tau_{xy}^2}$$

The equivalent tangential stresses at the various surfaces, in accordance with Birnie's equation, for use in conjunction with the maximum-strain theory of failure may be determined by:

On the surface at d_o for the outer member, $$s'_{to} = \frac{2p_c d_c^2}{d_o^2 - d_c^2}$$

On the surface at d_c for the outer member, $$s'_{tco} = p_c\left(\frac{d_o^2 + d_c^2}{d_o^2 - d_c^2} + \mu_o\right)$$

On the surface at d_c for the inner member, $$s'_{tci} = -p_c\left(\frac{d_c^2 + d_i^2}{d_c^2 - d_i^2} - \mu_i\right)$$

On the surface at d_i, $$s'_{ti} = \frac{-2p_c d_c^2}{d_c^2 - d_i^2}$$

FORCES AND TORQUES. The maximum axial force F_a required to assemble a force fit varies directly as the thickness of the outer member, the length of the outer member, the difference in diameters of the mating members, and the coefficient of friction. This force in pounds may be approximated by

$$F_a = f\pi dLp_c$$

The torque that can be transmitted by an interference fit without slipping between the hub and shaft can be estimated by

$$T = \frac{fp_c\pi d^2 L}{2}$$

where

F_a = axial load, lb

T = torque transmitted, in-lb

d = nominal shaft diameter, in.

f = coefficient of friction

L = length of external member, in.

p_c = contact pressure between the two members, psi

ASSEMBLY OF SHRINK FITS is often facilitated by heating the hub until it has expanded by an amount at least as much as the interference. The temperature change ΔT required to effect an increase δ in the inside diameter of the hub may be determined by

$$\Delta T = \frac{\delta}{\alpha\, d_i}$$

where

δ = diametral interference, in.

α = coefficient of linear expansion, per °F

ΔT = change in temperature, °F

d_i = initial diameter of the hole before expansion, in.

An alternate to heating the hub is to cool the shaft by means of a coolant such as dry ice.

2. In the design of rigid flange couplings, it is quite customary in this country to assume that the bolts are loosened in service and the capacity of the coupling is based, in part, upon the stresses set up in the bolts due to shearing of the bolts. The tightening effect of the bolts, with friction as the basis of power transmission, is generally neglected. However the purpose of this problem is to evaluate the capacity of a particular coupling based upon friction.

Assume that a flange coupling has the following specifications:

Number of bolts, 6

Size of bolts, $\frac{1}{2}$ in. diameter

Preloading of bolts, 5000 lb in each bolt

Inner diameter of contact, 7 in.

Outer diameter of contact, 8 in.

Speed of rotation of coupling, 300 rpm

Coefficient of friction, 0.15

Shaft diameter, 2 in.

Shaft material: SAE 1045, annealed with an ultimate tensile strength of 85,000 psi and a yield point in tension of 45,000 psi.

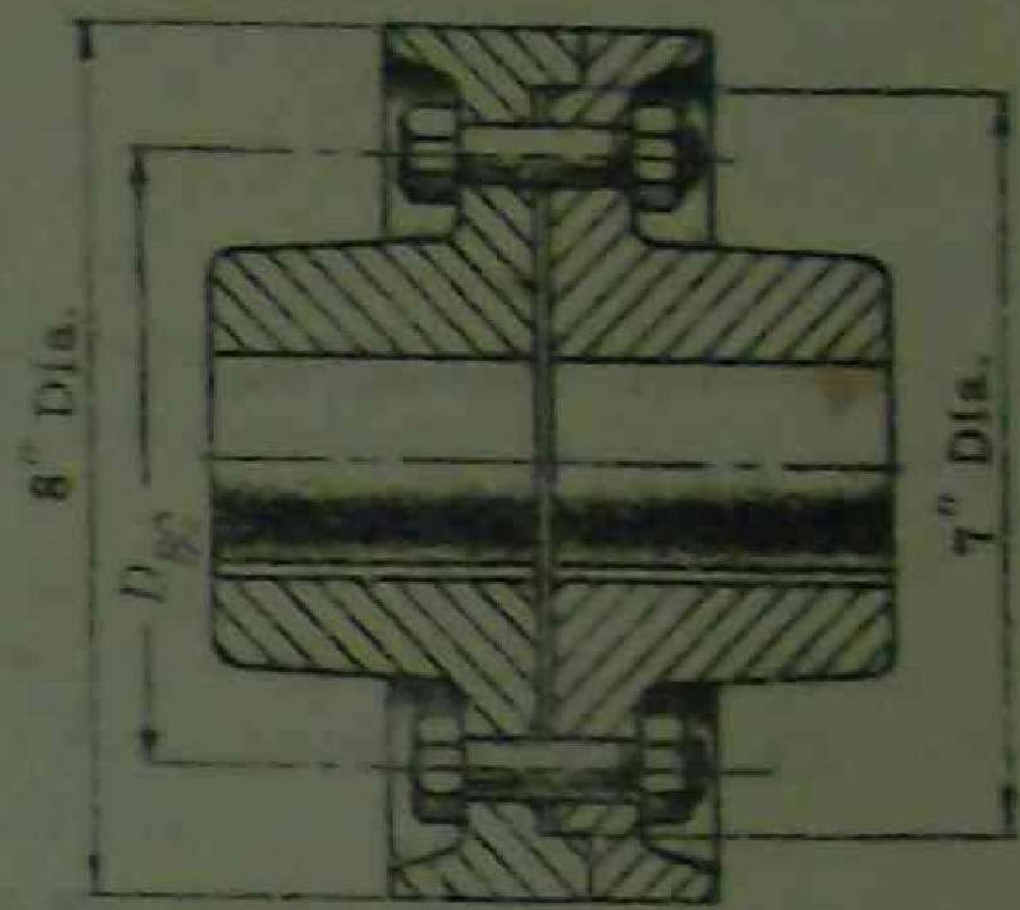


Fig. 10-2

The bolts are set in large clearance holes in the coupling. Refer to Fig. 10-2.

Determine:

(*1*) The maximum power capacity based upon friction such that slip occurs between faces of contact.

(*2*) Compare the shaft horsepower capacity with the friction horsepower capacity. Assume steady load conditions and that the shaft is in torsion only.

Solution:

(*a*) The torque capacity based on friction is (see chapter on Clutches)

$$M_t = FfR_f = 30{,}000(0.15)(3.75) = 16{,}900 \text{ in-lb}$$

where F = axial force caused by bolt loading = 30,000 lb

f = coefficient of friction

$$R_f = \text{friction radius} = \frac{2}{3}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right) = \frac{2}{3}\left(\frac{4^3 - 3.5^3}{4^2 - 3.5^2}\right) = 3.75 \text{ in.}$$

This assumes that the pressure is uniformly distributed.

$$\text{Friction horsepower} = \frac{M_t N}{63{,}024} = \frac{16{,}900(300)}{63{,}024} = 80.4 \text{ hp}$$

(*b*) Shaft torque capacity,

$$M_t = s_s \pi D^3/16 = 13{,}500(0.75)\pi D^3/16 = 15{,}900 \text{ in-lb}$$

where s_s is the smaller of $0.18(85{,}000) = 15{,}300$ psi and $0.3(45{,}000) = 13{,}500$ psi.

Note the factor 0.75 to take care of stress concentration.

$$\text{Shaft horsepower capacity} = \frac{M_t N}{63{,}024} = \frac{15{,}900(300)}{63{,}024} = 75.7 \text{ hp}$$

(*c*) For the data given, the coupling has greater horsepower capacity based on friction (80.4 hp) than the shaft capacity (75.7 hp).

Chapter 11

Keys, Pins, and Splines

KEYS are used to prevent relative motion between a shaft and the connected member through which torque is being transmitted. Even though gears, pulleys, etc., are assembled with an interference fit, it is desirable to use a key designed to transmit the full torque.

COMMON TYPES OF KEYS are the square key as shown in Fig. 11-1 (*a*), the flat key as shown in Fig. 11-1 (*b*), the Kennedy key as shown in Fig. 11-1 (*c*), and the Woodruff key as shown in Fig. 11-1 (*d*).

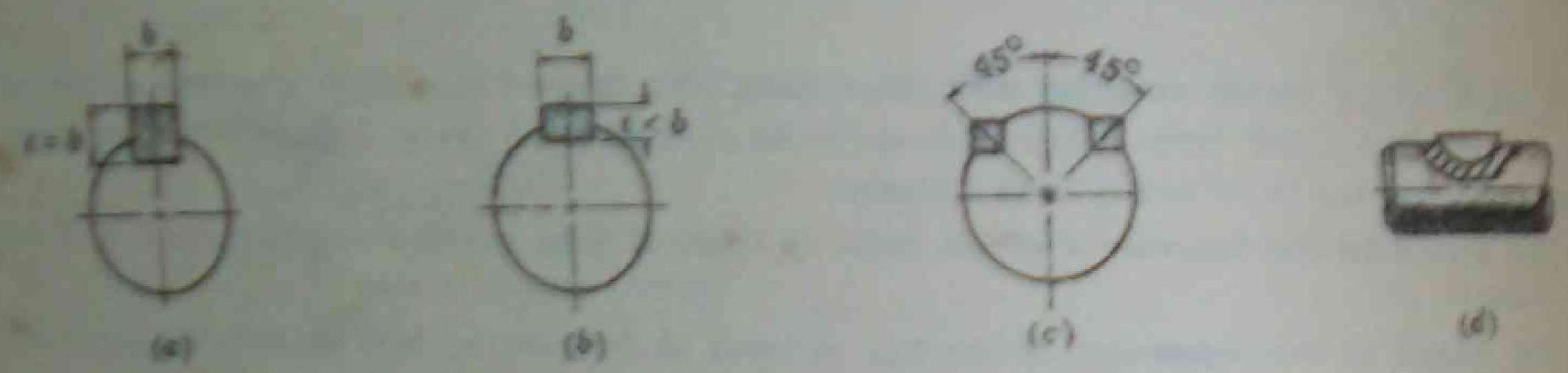


Fig. 11-1

The width of the square and the flat key is usually one fourth the diameter of the shaft. These keys may be either straight or tapered approximately $\frac{1}{8}$ in. per ft. A Gib-Head key is shown in Fig. 11-2. Feather keys and splines are used when it is necessary to have relative axial motion between the shaft and mating member. ASME and ASA standards for key and spline dimensions are available.

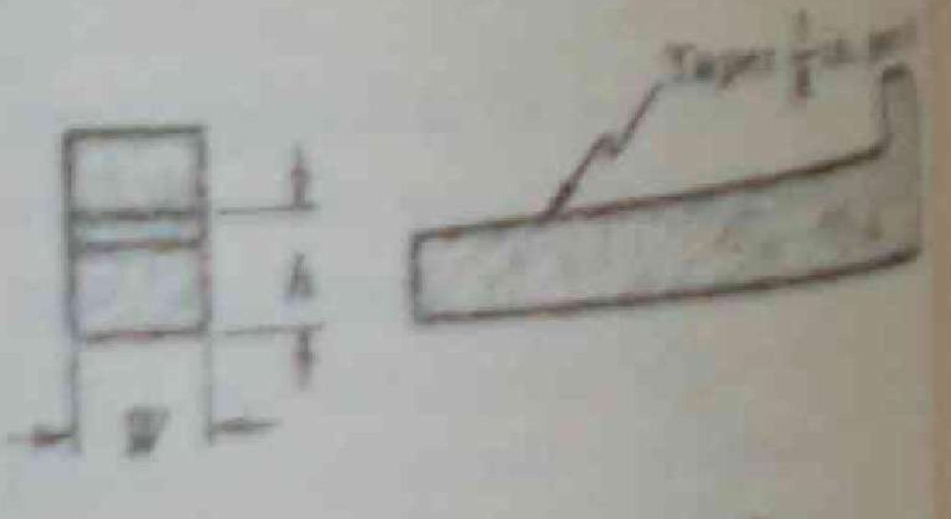


Fig. 11-2

DESIGN OF SQUARE AND FLAT KEYS may be based on the shear and compressive stresses induced in the key as a result of the torque being transmitted. The forces acting on the key are shown in Fig. 11-3. The forces F' act as a resisting couple to prevent the key from tending to roll in the fitted keyway. The exact location of the force F is not known and it is convenient to assume that it acts tangent to the surface of the shaft. This force produces both shear and compression stresses in the key.

Resistance to the shaft torque T may then be approximated by $T = Fr$, where r is the radius of the shaft. The shearing stress s_s in the key is

$$s_s = \frac{F}{bL} = \frac{Fr}{bLr} = \frac{T}{bLr}$$

where L is the length of the key.

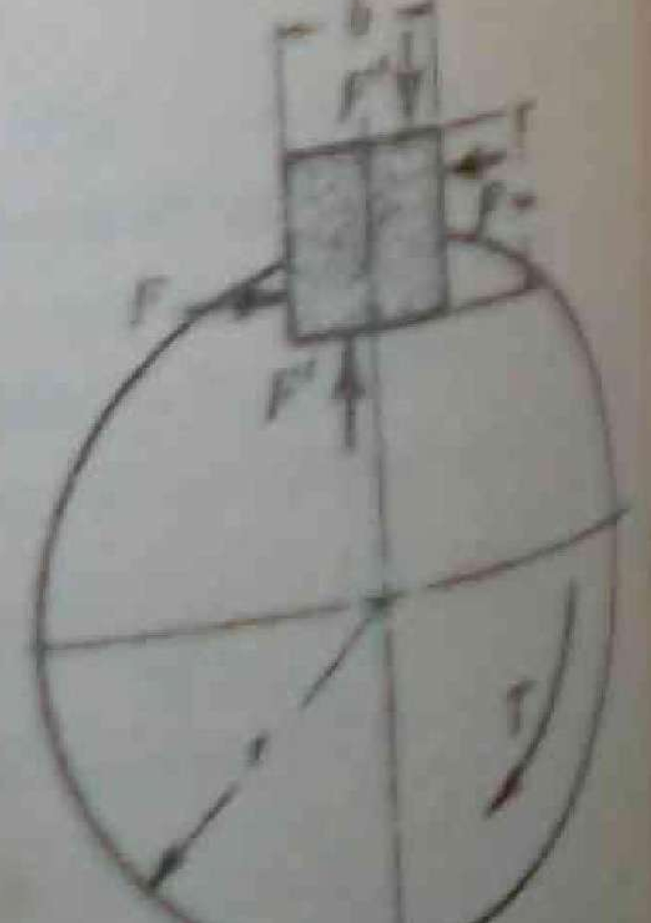


Fig. 11-3

The shaft torque that the key can sustain from the standpoint of <u>shear is</u>

$$\boxed{T_s = s_s bLr}$$

The compressive stress s_c in the key is

$$s_c = \frac{F}{(t/2)L} = \frac{Fr}{(t/2)Lr} = \frac{T}{(t/2)Lr}$$

The shaft torque that the key can sustain from the standpoint of <u>compression is</u>

$$\boxed{T_c = s_c(t/2)Lr}$$

A square key can sustain the same shaft torque from the standpoint of shear as it can from the standpoint of compression. This is proved as a solved problem by equating the two torque equations and making use of the approximate relation $s_c = 2s_s$ for ductile steels. On this same basis, flat keys which are wider than they are deep will fail in compression, and feather keys which are deeper than they are wide will fail in shear.

PINS are used in knuckle joints which connect two rods or bars loaded in either tension or compression as shown in Fig. 11-4 (*a*). An excessive load F may cause the joint to fail due to any of the following induced stresses.

1. Tensile stress in the rod:

$$s_t = \frac{4F}{\pi D^2}$$

2. Tensile stress in the net area of the eye, see Fig. 11-4(*b*):

$$s_t = \frac{F}{(d_o - d)b}$$

3. Shear stress in the eye due to tear out, see Fig. 11-4(*c*):

$$s_s = \frac{F}{b(d_o - d)} \quad \text{approx.}$$

4. Tensile stress in the net area of the clevis or fork:

$$s_t = \frac{F}{(d_o - d)2a}$$

5. Shear stress in the fork due to tear out:

$$s_s = \frac{F}{2a(d_o - d)} \quad \text{approx.}$$

6. Compressive stress in the eye due to bearing pressure of the pin:

$$s_c = \frac{F}{db}$$

7. Compressive stress in the fork due to bearing pressure of the pin:

$$s_c = \frac{F}{2da}$$

8. Shear stress in the pin:

$$s_s = \frac{F}{A} = \frac{2F}{\pi d^2}$$

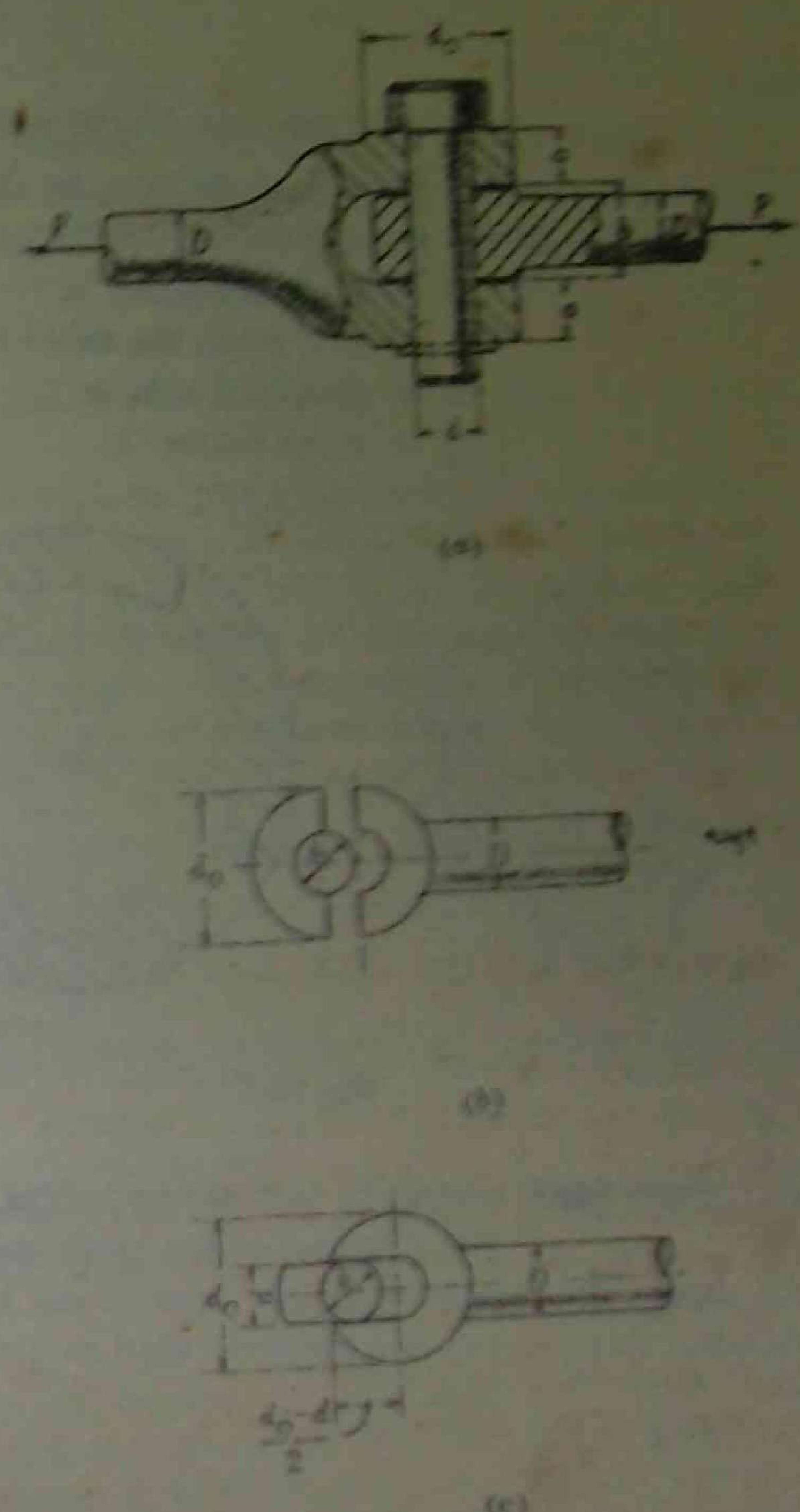


Fig. 11-4

9. Bending stress in the pin, based on assumption that the pin is supported and loaded as shown in Fig.11-5. The maximum bending moment M_b occurs at the center of the pin. $M_b = Fb/8$, $I = \pi d^4/64$, $c = d/2$, and

$$s_b = \frac{Mc}{I} = \frac{4Fb}{\pi d^3}$$

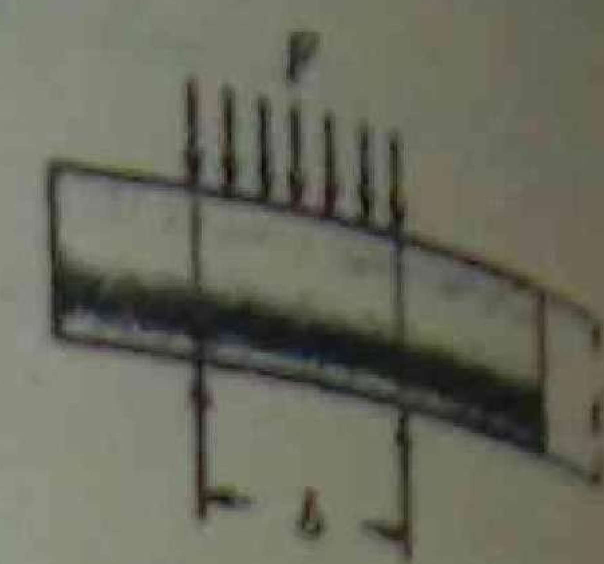


Fig. 11-5

10. Compressive stress in the pin due to the eye: $s_c = \dfrac{F}{db}$

11. Compressive stress in the pin due to the fork: $s_c = \dfrac{F}{2ad}$

SPLINED CONNECTIONS as shown in Fig.11-6 below are used to permit relative axial motion between the shaft and hub of the connected member. The splines are keys made integral with the shaft and usually consist of four, six, or ten in number. The keyways are broached in the hub to the required fit. The splines are usually made with straight sides or cut with an involute profile. When there is relative axial motion in a splined connection, the side pressure on the splines should be limited to about 1000 psi. The torque capacity of a splined connection is

$$T = pAr_m$$

where

- p = permissible pressure on the splines, <1000 psi
- A = total load area of splines, sq in.
  - = $\frac{1}{2}(D-d)(L)$(number of splines), sq in.
    - D = shaft diameter, in.
    - d = D − twice the depth of the spline, in.
    - L = length of hub, in.
    - r_m = mean radius, in.

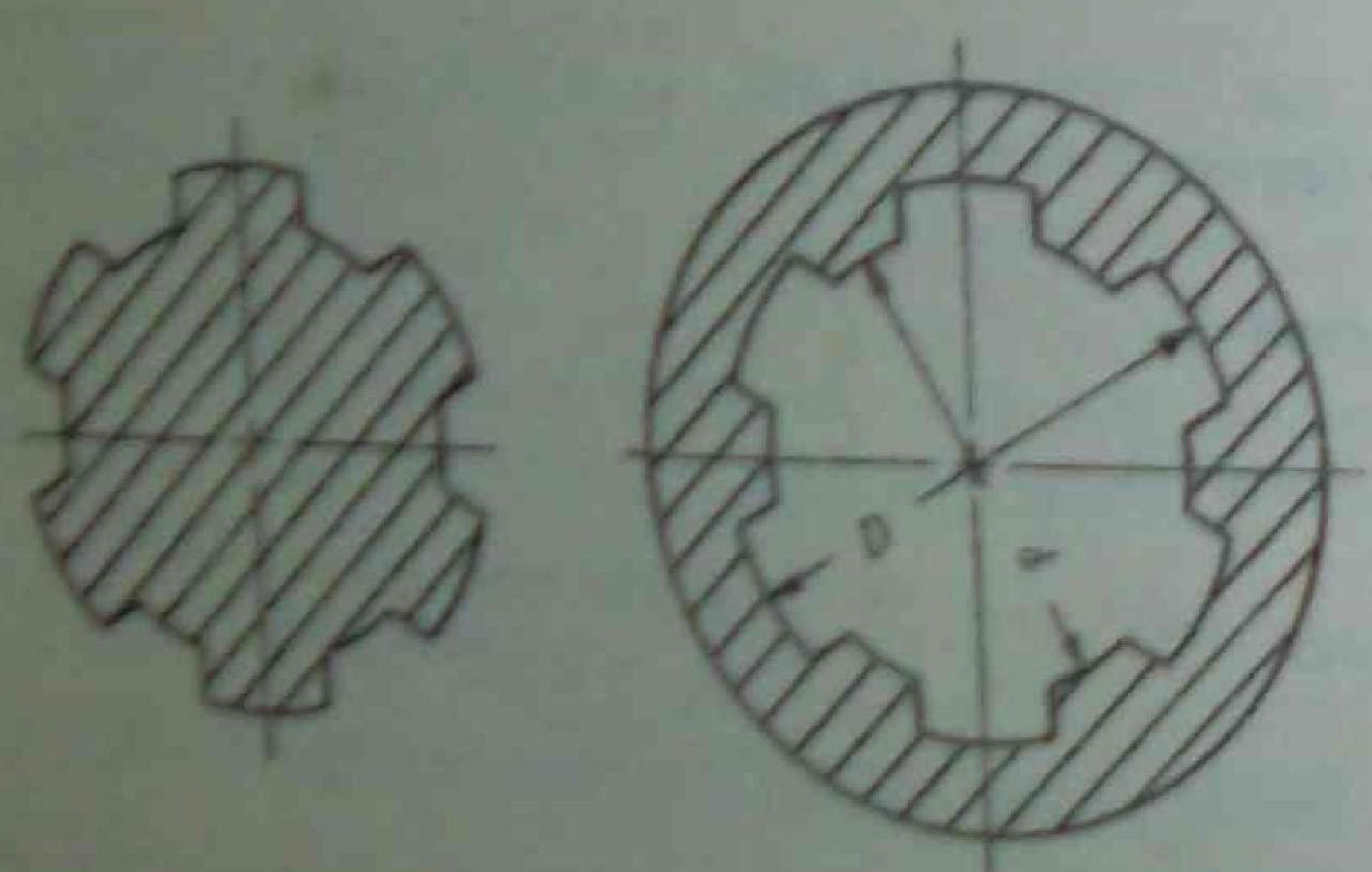


Fig. 11-6

Fig. 11-7

TAPERED PINS similar to that shown in Fig.11-7 above are frequently used to key hubs to shafts. The diameter of the large end of the pin is usually about one-fourth the diameter of the shaft. The capacity of this type of pin key connection is determined by the two shear areas of the pin. The pin key is sometimes used as a shear pin.

SOLVED PROBLEMS

1. Prove that a square key is equally strong in shear and compression.

Solution:

From the standpoint of compression a key can sustain the following torque: $T_c = s_c(t/2)Lr$.

From the standpoint of shear a key can sustain the following torque: $T_s = s_s bLr$.

Equating the two torque equations to establish equal strength in shear and compression, and assuming that $s_c = 2s_s$, we have $T_c = T_s$, $2s_s(t/2)Lr = s_s bLr$, and $t = b$ (a square key).

2. Determine the required length of a square key if the key and shaft are to be of the same material and of equal strength.

Solution:

The torque that a shaft of diameter d can transmit allowing for a 25% reduction due to stress concentration is $T = 0.75\pi d^3 s_s/16$. Equating this torque to the torque that a square key can sustain from the standpoint of shear, we have

$$0.75\pi d^3 s_s/16 = s_s bLr$$

Substitute $d/4$ for b and solve for $L = 1.18d$; use $L = 1.25d$.

3. A square key is to be used to key a gear to a $1\frac{7}{16}$ in. diameter shaft. The hub length of the gear is $2\frac{1}{2}$ in. Both shaft and key are to be made of the same material, having an allowable shear stress of 8000 psi. What are the minimum dimensions for the sides of the square key if 3490 in-lb of torque is to be transmitted?

Solution:

Equate the expression for the torque that the key can sustain from the standpoint of shear to 3490 in-lb and solve for b.

$rLbs_s = 3490$, $(1.4375/2)(2.5)(b)(8000) = 3490$, $b = 0.243$ in. Use a standard 0.25 in. square key.

4. A feather key is $\frac{1}{2}$ in. wide by $\frac{3}{8}$ in. deep and is to transmit 6000 in-lb of torque from a $1\frac{1}{2}$ in. diameter shaft. The steel key has an allowable stress in tension and compression of 16,000 psi and an allowable stress in shear of 8000 psi. Determine the required length of the key.

Solution:

Since the key is wider than it is deep, it will fail in compression before it will fail in shear.

$s_c(t/2)Lr = 6000$, $(16{,}000)(0.375/2)(L)(0.75) = 6000$, $L = 2.66$ in. Use $L = 2\frac{3}{4}$ in.

5. If the key in Problem 4 had been 3/8 in. wide and $\frac{1}{2}$ in. deep, what would have been the required length for the same load and material?

Solution:

In this case the key is weaker in shear than it is in compression.

$s_s bLr = 6000$, $(8000)(0.5)(L)(0.75) = 6000$, $L = 2$ in.

6. A pin in a knuckle joint as shown in Fig. 11-4(a) is subjected to an axial load of 20,270 lb. Assume that the thickness of the eye is to be 1.5 times the diameter of the pin. The allowable stress of the material in tension and compression due to bending is 9000 psi, and the allowable stress in shear is 4500 psi. The allowable bearing stress is 3000 psi. Determine the required pin diameter.

Solution:

Check the pin for (a) bending, (b) shear, and (c) bearing.

Chapter 12

Power Screws and Threaded Fasteners

POWER SCREWS provide a means for obtaining large mechanical advantage in such applications as screw jacks, clamps, presses, and aircraft control-surface actuators. Occasionally they are used in reverse for such applications as push drills.

THREADED FASTENERS include through bolts, stud bolts, cap screws, machine screws, set screws, and a variety of more special devices using the screw principle.

TERMINOLOGY OF SCREW THREADS is illustrated in Figure 12-1. Thread form is ordinarily described in axial section. The square and the Acme forms are commonly used for power screws (Fig.12-2 below). For threaded fasteners, the Unified and American standard thread form has the basic shape and proportions shown in Fig.12-3 below. This basic shape has maximum metal content. Variations for different classes of fit are made in the direction of greater metal removal. For detailed tables of standard sizes, thread series, and information on classes of fit see any standard Machine Design text or Mechanical Engineers handbook.

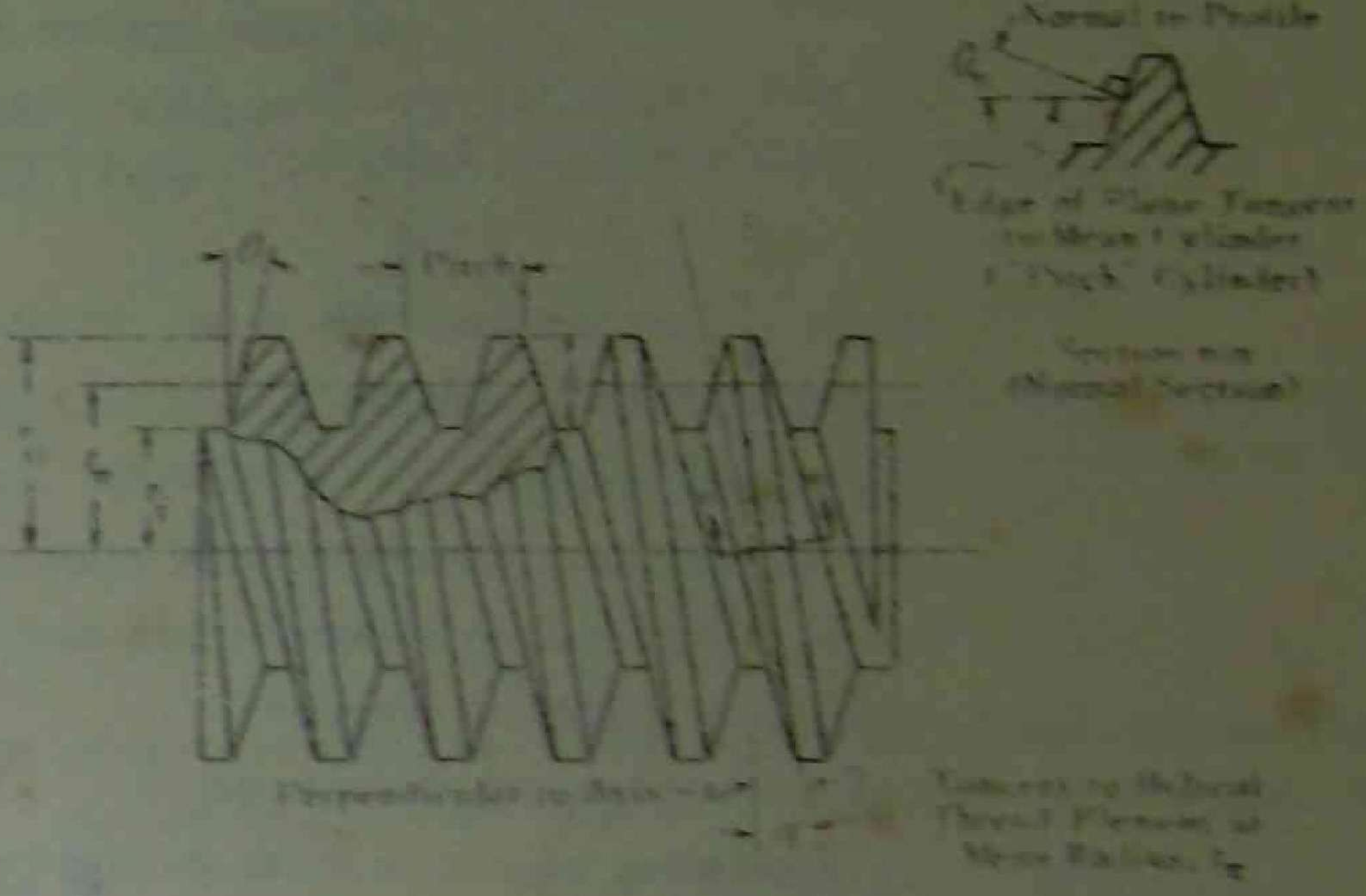


Fig. 12-1

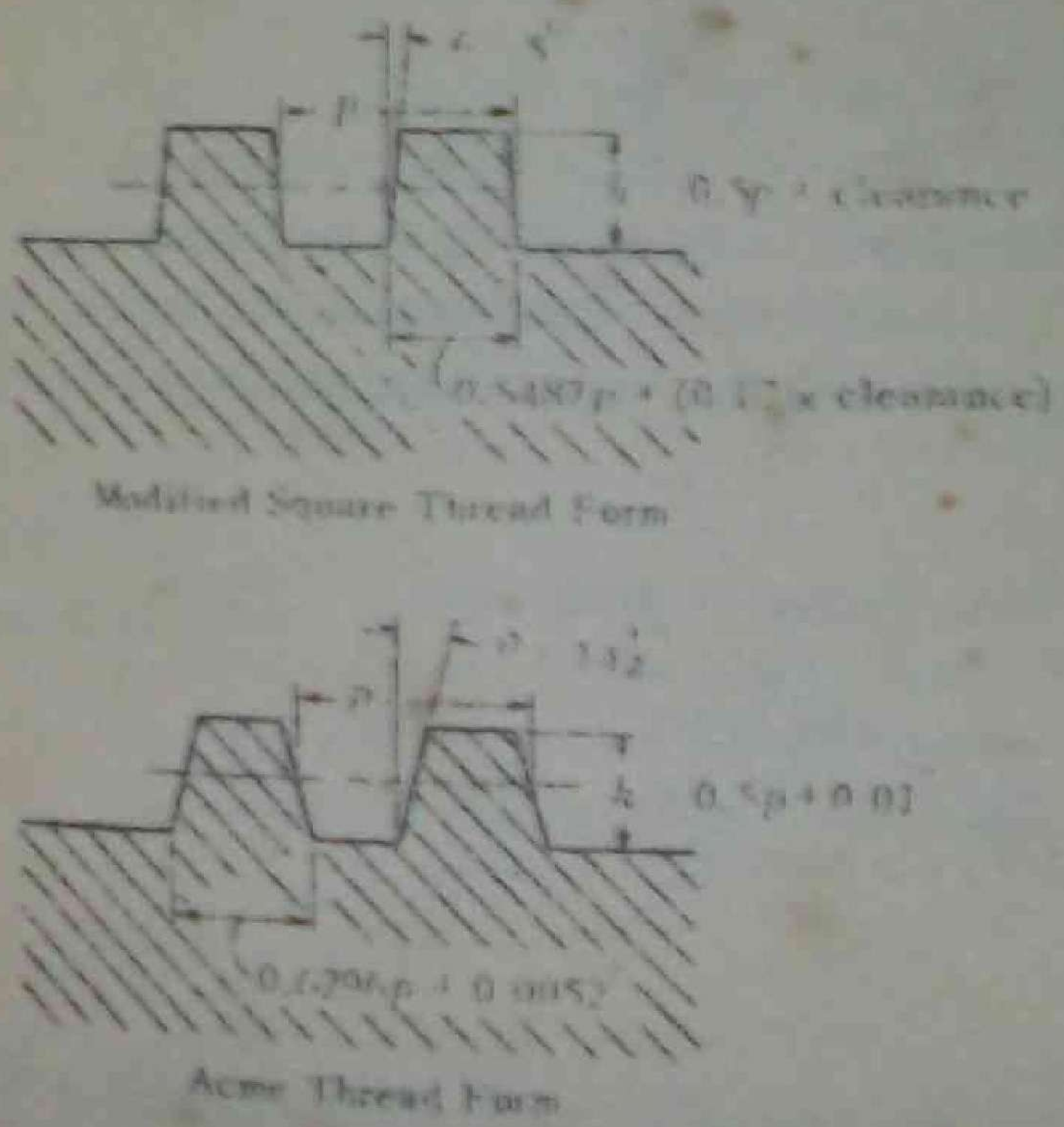


Fig. 12-2

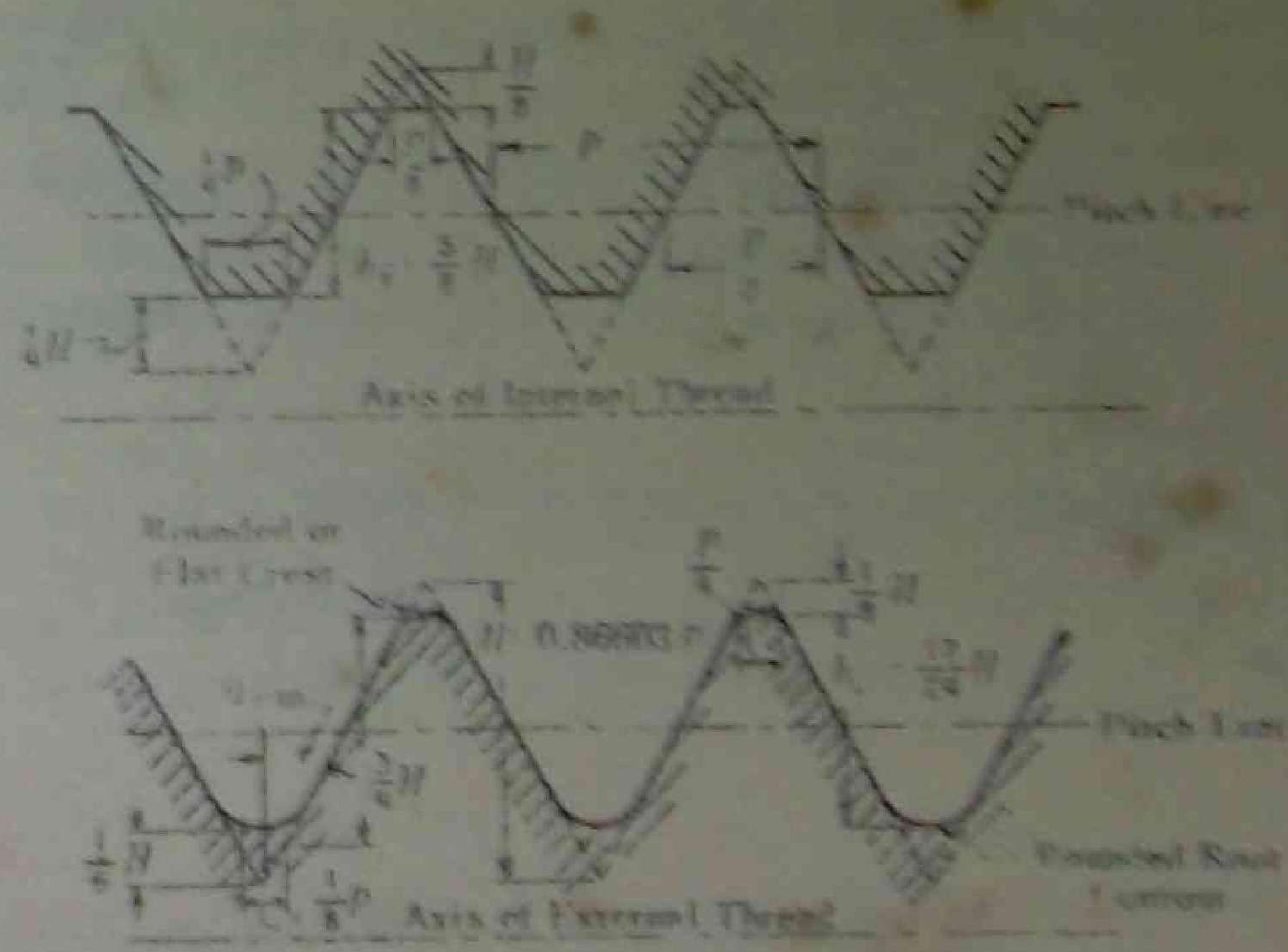


Fig. 12-3

Pitch is the distance from a point on one thread to the corresponding point on the next adjacent thread, measured parallel to the axis.

Lead is the distance the screw would advance relative to the nut in one rotation. For a single-thread screw, lead is equal to pitch. For a double-thread screw, lead is equal to twice the pitch, etc.

Helix angle α is related to the lead and the mean radius r_m by the equation

$$\tan\alpha = \frac{\text{lead}}{2\pi r_m}$$

In some calculations we shall make use of the angle θ_n measuring the slope of the thread profile in the normal section. This is related to the angle θ in the axial section and to the helix angle as follows:

$$\tan\theta_n = \tan\theta\cos\alpha$$

Note: Where $\cos\theta_n$ appears in the equations to follow, it is frequently replaced by $\cos\theta$. This yields an approximate equation but, for the usual small values of α, introduces no great error.

TURNING MOMENT AND AXIAL LOAD are related to each other through the following equation for advance **against load** (or *raising* the load):

$$T = W\left[r_m\left(\frac{\tan\alpha + f/\cos\theta_n}{1 - f\tan\alpha/\cos\theta_n}\right) + f_c r_c\right]$$

where

- T = torque applied to turn screw or nut, whichever is being rotated
- W = load parallel to screw axis
- r_m = mean thread radius
- r_c = effective radius of rubbing surface against which load bears, called *collar radius*
- f = coefficient of friction between screw and nut threads
- f_c = coefficient of friction at *collar*
- α = helix angle of thread at mean radius
- θ_n = angle between tangent to tooth profile (on the loaded side) and a radial line, measured in plane normal to thread helix at mean radius.

The torque required to advance the screw (or nut) **in the direction of the load** (or *lowering* the load) is

$$T = W\left[r_m\left(\frac{-\tan\alpha + f/\cos\theta_n}{1 + f\tan\alpha/\cos\theta_n}\right) + f_c r_c\right]$$

This torque may be either positive or negative. If positive, work must be done to advance the screw. If negative, the meaning is that, for equilibrium, the torque must retard rotation, i.e. the axial load alone will cause rotation (the push drill situation). In this case the screw is said to be overhauling.

EFFICIENCY OF A SCREW MECHANISM is the ratio of work output to work input.

$$\text{Efficiency} = \frac{100(W)(\text{lead})}{2\pi T}\,\% = \frac{100\tan\alpha}{\left(\dfrac{\tan\alpha + f/\cos\theta_n}{1 - f\tan\alpha/\cos\theta_n}\right) + \dfrac{f_c r_c}{r_m}}\,\%$$

STRESSES IN THE THREAD are estimated by considering the thread to be a short cantilever beam projecting from the root cylinder. (See Fig. 12-4 below.) The beam load is taken to be the axial screw load W, assumed concentrated at the mean radius, i.e. at one-half the thread depth h. The beam width is the length of thread (measured at mean radius) subject to the

With these assumptions the bending stress at the root of the thread is, very nearly,

$$s_b = \frac{3Wh}{2\pi n r_m b^2}$$

and the mean transverse shear stress is

$$s_s = \frac{W}{2\pi n r_m b}$$

where n is the number of thread turns subject to load and b is the width of thread section at the root.

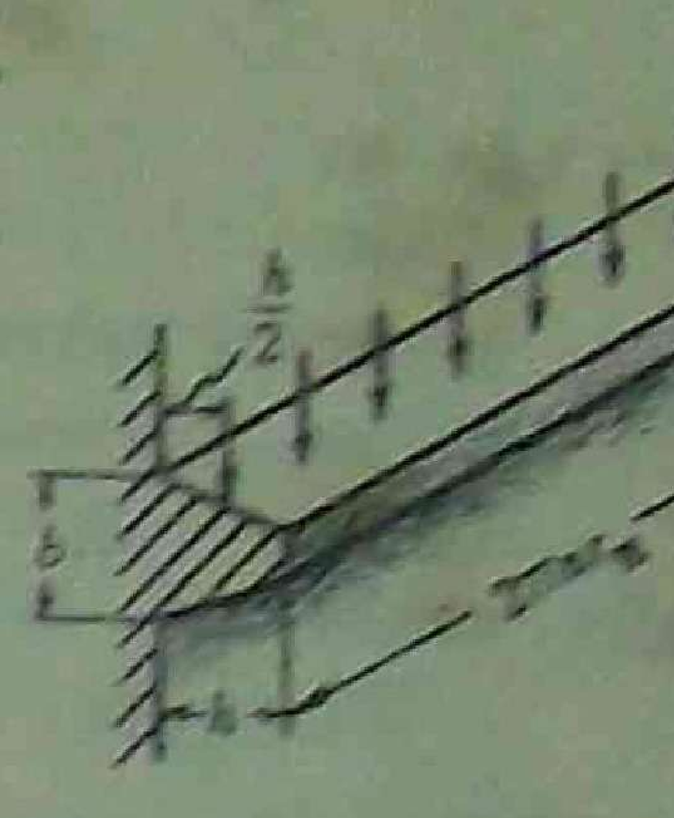


Fig. 12-4

The stress picture at the junction of the thread and root cylinder is actually very complicated, and the above expressions are only rough approximations which serve as design guides. In place of r_m in these expressions, many designers will use r_i for the screw and r_o for the nut, a somewhat better approximation in that it recognizes the nut threads to be less likely to *strip* than the screw threads.

BEARING PRESSURE between the surfaces of screw and nut threads may be a critical factor in design, especially for power screws. It is given approximately by

$$p = \frac{W}{2\pi n r_m h}$$

This estimate will be on the low side because (*1*) clearances between root and crest of internal and external threads mean that load is not carried over full depth h and (*2*) load is not uniformly distributed over thread length.

STRESSES IN THE ROOT CYLINDER of the screw may be estimated by considering loads and torques carried by the bare cylinder (neglecting strengthening effect of thread). The torsional shear stress is

$$s_s = \frac{2T}{\pi r_i^3}$$

where r_i is the root radius of the screw. T is the appropriate torque, i.e. the torque to which the section under consideration is subjected. This may be the total applied torque, only the collar friction torque, or only the *screw* torque (total minus collar friction torque). Each case must be examined carefully to see which applies.

The direct stress, which may be either tensile or compressive, is

$$s_n = \frac{W}{\text{root area}} = \frac{W}{\pi r_i^2}$$

A modification of the above formula is sometimes used in calculations on threaded fastenings to account approximately for the strengthening effect of the threads. Basically the modification consists of presuming the cylinder to be larger in radius than it really is. (See Chapter 13.) Then

$$s_n = \frac{W}{\text{stress area}}$$

Stress areas, as well as root areas, are tabulated in many textbooks and handbooks.

SOLVED PROBLEMS

1\. The screw in Fig. 12-5 below is operated by a torque applied to the lower end. The nut is loaded and prevented from turning by guides. Assume friction in the ball bearing to be negligible. The screw has a 2 in. outside diameter and a triple Acme thread, 3 threads per inch. Thread coefficient of friction is 0.15. Determine the load which could be raised by a torque T of 400 in-lb.

Solution:

$$T = W[r_m(\frac{\tan\alpha + f/\cos\theta_n}{1 - f\tan\alpha/\cos\theta_n}) + f_c r_c]$$

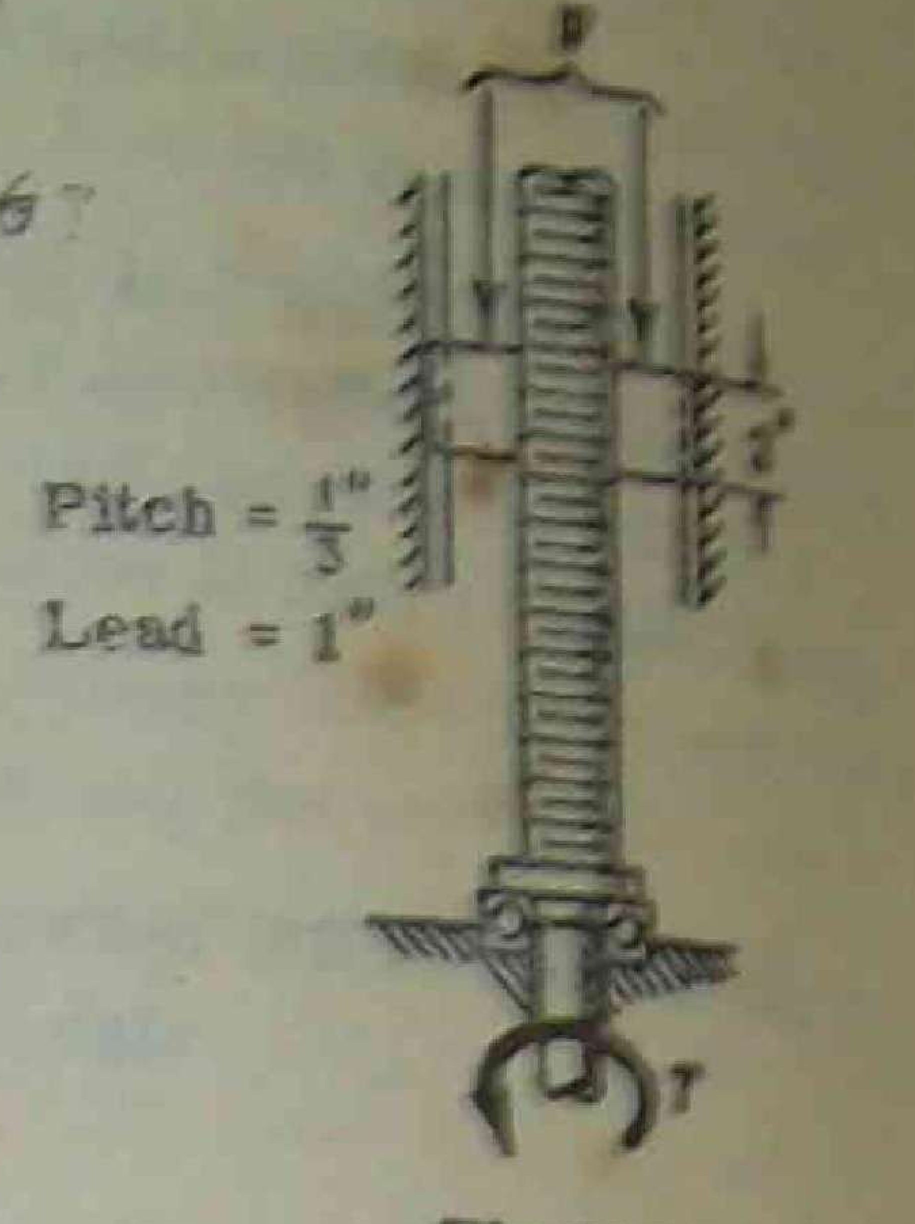


Fig. 12-5

where:

thread depth $= 0.18$ in.

$r_m = 1.00 - 0.18/2 = 0.91$ in.

$$\tan\alpha = \frac{\text{lead}}{2\pi r_m} = \frac{1.00}{2\pi(0.91)} = 0.175$$

$\alpha = 9.92^\circ$

$\theta = 14.5^\circ$ for Acme thread

$\tan\theta_n = (\tan\theta)(\cos\alpha)$

$= (\tan 14.5^\circ)(\cos 9.92^\circ) = 0.255$

$\theta_n = 14.2^\circ$

Note that the difference between θ_n and θ is so slight that we might as well use θ. Then

$$400 = W[0.91(\frac{0.175 + 0.15/0.968}{1 - (0.15)(0.175)/0.968}) + 0] \quad \text{or} \quad W = 1290 \text{ lb}$$

2\. Would the screw of Problem 1 be overhauling?

Solution:

The screw will be overhauling if the torque in the following equation is negative.

$$T = W[r_m(\frac{-\tan\alpha + f/\cos\theta_n}{1 + f\tan\alpha/\cos\theta_n}) + f_c r_c]$$

Since $f_c = 0$ for this problem, T will be negative if $(-\tan\alpha + f/\cos\theta_n)$ is negative.

From Problem 1, $\tan\alpha = 0.175$ and $f/\cos\theta_n = 0.15/0.968 = 0.155$. Hence the screw will overhaul, i.e. to prevent screw rotation when a load W is applied, a holding torque must be applied.

3\. For the screw of Problem 1 determine the average bearing pressure between the screw and nut thread surfaces.

Solution:

$$p = \frac{W}{2\pi n r_m h} = \frac{1290}{2\pi(6)(0.91)(0.18)} = 210 \text{ psi}$$

where $n = \dfrac{\text{nut length}}{\text{pitch}} = \dfrac{2}{1/3} = 6$ thread turns sharing the load.

4\. Derive the equation for torque T required to advance a screw against a load W.

Solution:

Refer to Fig. 12-6 below. The total normal force exerted by the threads of the nut against the threads of the screw is F_n. This is distributed over the length of thread in engagement and over the thread depth, but for the purposes of this analysis may be considered concentrated at a point at the mean screw radius, r_m.

Fig. 12-6

The normal force vector F_n appears in true length in the normal section. Component $F_n \cos\theta_n$ is tangent to the pitch cylinder and at angle α (helix angle) with the screw axis. Component $F_n \sin\theta_n$ is radial.

The friction force is fF_n, acting along the thread helix.

Also acting on the screw is the axial load W, the collar friction torque $Wf_c r_c$, and the applied torque T.

Summing forces parallel to the screw axis yields

$$W - F_n \cos\theta_n \cos\alpha + fF_n \sin\alpha = 0$$

Summing moments around the screw axis yields

$$T - F_n r_m \cos\theta_n \sin\alpha - fF_n r_m \cos\alpha - Wf_c r_c = 0$$

Eliminating F_n between these two equations yields

$$T = W\left[r_m\left(\frac{\tan\alpha + f/\cos\theta_n}{1 - f\tan\alpha/\cos\theta_n}\right) + f_c r_c\right]$$

5. The following data apply to the C-clamp of Fig. 12-7.

American Standard Threads
13 threads per inch (single-threaded)
Outside diameter = $\frac{1}{2}$ in.
Root diameter = 0.4001 in.
Root area = 0.1257 in^2
Coefficient of thread friction = 0.12 (= f)
Coefficient of collar friction = 0.25 (= f_c)
Mean collar radius = 0.25 in.
Load W = 1000 lb
Operator can comfortably exert a force of 20 lb at the end of the handle.

(a) What length of handle, L, is needed?

(b) What is the maximum shear stress in the body of the screw and where does this exist?

(c) What is the bearing pressure P on the threads?

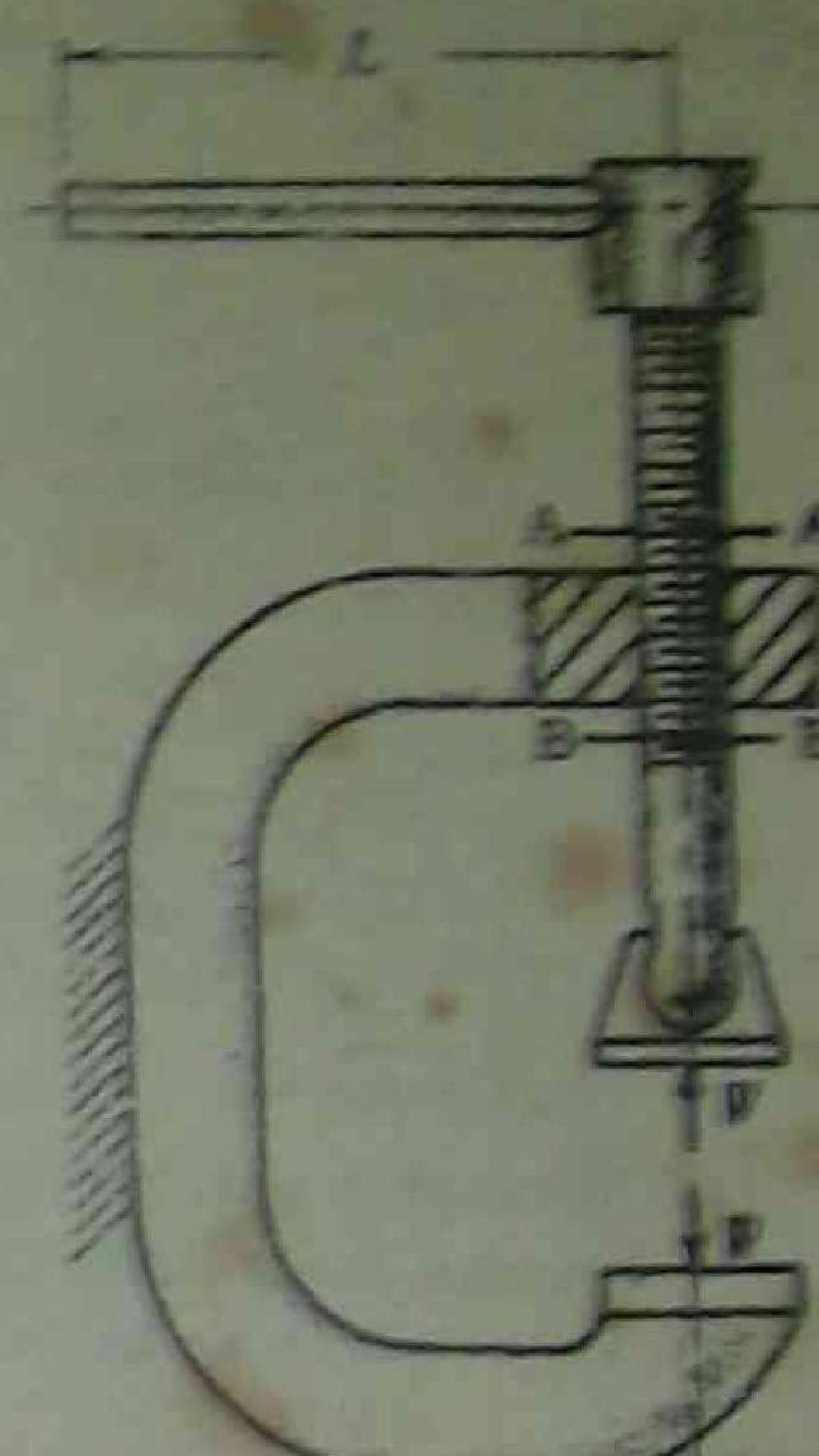


Fig. 12-7

Solution:

(a) The torque required is

$$T = W\left[r_m\left(\frac{\tan\alpha + f/\cos\theta_n}{1 - f\tan\alpha/\cos\theta_n}\right) + f_c r_c\right]$$

where $r_m = \frac{1}{4}(0.5000 + 0.4001) = 0.225$ in., $\tan\alpha = \dfrac{\text{lead}}{2\pi r_m} = \dfrac{1/13}{2\pi(0.225)} = 0.0544$.

1. What are the values of allowance, hole tolerance, and shaft tolerance for the following dimensions of mated parts according to the basic hole system?

Hole	1.5000"	**Shaft**	1.4988"
	1.5009"		1.4978"

Solution:

Hole 1.5000" to $1.5'' + t_h$ $\quad t_h = 0.0009''$

Shaft $1.5'' - a$ $\quad a = 0.0012''$ to $1.5'' - a - t_s$ $\quad t_s = 0.0010''$

2. A 3" shaft rotates in a bearing. The tolerance for both shaft and bearing is 0.003" and the required allowance is 0.004." Dimension both the shaft and bearing bore in accordance with the basic hole standard.

Solution:

Hole $d = 3.000''$, $d + t_h = 3.003''$

Shaft $d - a = 2.996''$, $d - a - t_s = 2.993''$

3. A medium force fit on a 3 inch shaft requires a hole tolerance of 0.009 in., a shaft tolerance of 0.009 in., and an average interference of 0.0015 in. Give the proper hole and shaft dimensions in accordance with the basic hole standard.

Solution:

Hole $d = 3.0000''$, $d + t_h = 3.0009''$

Shaft $d + i = 3.0015''$, $d + i + t_s = 3.0024''$

4. (a) What is the difference in the type of assembly generally used in running fits and interference fits?
(b) If a medium force fit (0.0015 in. interference) is desired, which axle should be fitted to each car wheel in the following group?

Wheel	**A**	**B**	**C**
Hole Diameter	3.0009"	3.0005"	3.0000"
Axle	**A′**	**B′**	**C′**
Diameter	3.0015"	3.0020"	3.0024"

Solution:

(a) Running fits are strictly interchangeable while interference fits require selective assembly.
(b) For selective assembly A′ should be mated with **C**, **B′** with **B**, and **C′** with **A**.

5. Give the dimensions for the hole and shaft for the following: (a) a ½ inch electric motor sleeve bearing, (b) a medium force fit on an 8 inch shaft, (c) a 2 inch sleeve bearing on the elevating mechanism of a road grader.

Solution:

(a) A class 2, free fit, would be suitable for an electric motor sleeve bearing.
Allowance = $0.0014 \times 0.5^{2/3} = 0.0009$ in.
Tolerance (shaft and hole) = $0.0013 \times 0.5^{1/3} = 0.0010$ in.
Hole dimen. = d to $d + t_h$ = 0.5000 to 0.5010 in.
Shaft dimen. = $d - a$ to $d - a - t_s$ = 0.4991 to 0.4981 in.

(b) Interference = $0.0005 \times 8 = 0.0040$ in.
Tolerance (shaft and hole) = $0.0006 \times 8^{1/3} = 0.0012$ in.
Hole dimen. = d to $d + t_h$ = 8.0000 to 8.0012 in.
Shaft dimen. = $d + i$ to $d + i + t_s$ = 8.0040 to 8.0052 in.

(c) A class 1, loose fit, would be suitable.
Allowance = $.0025 \times 2^{2/3} = 0.004$ in.
Tolerance (shaft and hole) = $.0025 \times 2^{1/3} = 0.0031$ in.
Hole dimensions = 2.000 to 2.003 in.
Shaft dimensions = 1.996 to 1.993 in.

We shall take $\theta_n = \theta = 30^\circ$, since the helix angle is so small. Then

$$T = 1000\left[0.225\left(\frac{0.0544 + 0.12/0.866}{1-(0.12)(0.0544)/0.866}\right) + (0.25)(0.25)\right]$$

$$= 43.8 \text{ (screw torque)} + 62.5 \text{ (collar torque)} = 106.3 \text{ in-lb}$$

To develop this torque with a 20 lb force, we need $L = 106.3/20 = 5.32$ in.

(b) Section A-A, just above the nut, is subjected to torque and bending. Section B-B, just below the nut, is subjected to torque and direct compressive load. It will be necessary to check both sections for maximum shear stress.

At A-A

$$\text{Torsional shear stress, } s_s = \frac{Tr}{J} = \frac{(106.3)(0.200)}{0.00251} = 8470 \text{ psi}$$

where $T = 106.3$ in-lb (from above), $r = r_i = 0.200$ in., $J = \frac{1}{2}\pi r_i^4 = 0.00251$ in^4.

$$\text{Bending stress, } s_t = \frac{M_b c}{I} = \frac{(120)(0.200)}{0.00126} = 19{,}100 \text{ psi}$$

where $M_b = (20)(6) = 120$ in-lb, $c = r_i = 0.200$ in., $I = \frac{1}{4}\pi r_i^4 = 0.00126$ in^4

$$\text{Maximum shear stress, } \tau(\max) = \sqrt{(\tfrac{1}{2}s_t)^2 + s_s^2} = 12{,}750 \text{ psi}$$

At B-B

This section is subjected to the collar friction torque $W f_c r_c$. Hence the torsional shear stress is

$$s_s = \frac{(W f_c r_c) r_i}{J} = \frac{(1000)(0.25)(0.25)(0.200)}{0.00251} = 4970 \text{ psi}$$

The direct compressive stress is $s_c = \dfrac{W}{A} = \dfrac{1000}{0.1257} = 7960$ psi.

Hence the maximum shear stress is $\tau(\max) = \sqrt{(7960/2)^2 + (4970)^2} = 6370$ psi

Our conclusion, then, is that the maximum shear stress occurs at section A-A and is 12,750 psi.

(c)

$$P = \frac{W}{2\pi n r_m h} = \frac{1000}{2\pi(13)(0.225)(0.050)} = 1090 \text{ psi}$$

where $n = \dfrac{\text{nut length}}{\text{pitch}} = \dfrac{1}{1/13} = 13$ threads, and $h = r_o - r_i = 0.250 - 0.200 = 0.050$ in.

6. It is proposed to make a screw jack in accordance with the sketch of Fig. 12-8. Neither screw rotates. Outside screw diameter is 2 in. Thread is square (depth = ½ pitch), single thread, and coefficient of thread friction is estimated to be 0.15.

(a) What would be the efficiency of the jack?

(b) What load can be raised if the shear stress in the bodies of the screws is limited to 4000 psi? Assume torque applied to nut in such a way as to cause no bending stress in the lower screw.

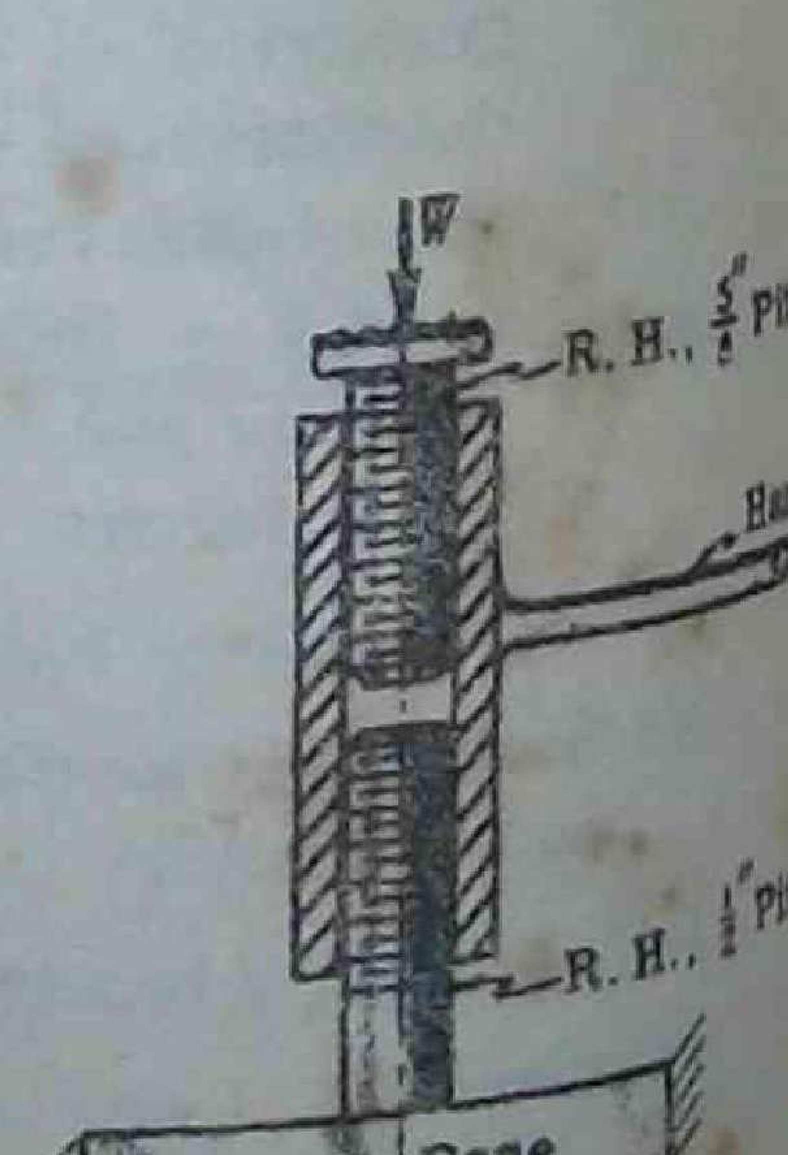


Fig. 12-8

Solution:

(a) A differential screw action is involved. In one turn of the nut the load is raised an amount equal to the difference in leads of the two screws. Hence the output work in one turn is

Output work per turn $= W$ (lead of upper screw − lead of lower screw)
$= W(\frac{5}{8} - \frac{1}{2}) = W/8$ in-lb

No collar friction is involved since there is no rotation of the screw to which the load is applied.

To turn the nut relative to the upper screw requires the following torque T'

$$T' = W\left[r'_m\left(\frac{\tan\alpha' + f/\cos\theta'_n}{1-f\tan\alpha'/\cos\theta'_n}\right)\right] = W\left[\frac{27}{32}\left(\frac{0.118 + 0.15}{1-(0.15)(0.118)}\right)\right] = 0.230W \text{ in-lb (upper screw)}$$

where $r'_m = r'_o - \dfrac{\text{thread depth}}{2} = 1-\dfrac{5}{32} = \dfrac{27}{32}$ in., $\tan\alpha' = \dfrac{\text{lead}}{2\pi r'_m} = \dfrac{5/8}{2\pi(27/32)} = 0.118$

To turn the nut relative to the lower screw requires the following torque T''. Note that in this case the nut advances in the direction of the load applied to the nut.

$$T'' = W\left[r''_m\left(\frac{-\tan\alpha'' + f/\cos\theta''_n}{1 + f\tan\alpha''/\cos\theta''_n}\right)\right] = W\left[\frac{7}{8}\left(\frac{-0.0910 + 0.15}{1 + (0.15)(0.0910)}\right)\right] = 0.051W \text{ in-lb (lower screw)}$$

where $r''_m = r''_o - \dfrac{\text{thread depth}}{2} = 1-\dfrac{1}{8} = \dfrac{7}{8}$ in. $\tan\alpha'' = \dfrac{\text{lead}}{2\pi r''_m} = \dfrac{1/2}{2\pi(7/8)} = 0.091($

and we take $\theta''_n = \theta'' = 0°$.

Hence the total torque T to be applied to the nut is $T = T' + T'' = 0.281W$ in-lb.

$$\text{Efficiency} = \frac{\text{output work per turn}}{\text{input work per turn}} = \frac{W/8}{2\pi(0.281W)} = 0.071 = 7.1\%$$

(b) The upper screw will be critical, since it is subjected to the larger torque and has the smaller root area. There is a direct compressive stress s_c and a torsional shear stress s_s to be taken into account.

$$s_c = \frac{W}{\text{root area}} = \frac{W}{\pi r_i^2} = \frac{W}{\pi(11/16)^2} = 0.674W$$

where $r_i = r_o - \text{thread depth} = 1-\frac{5}{16} = \frac{11}{16}$ in.

$$s_s = \frac{Tr}{J} = \frac{(0.230W)(11/16)}{0.350} = 0.452W$$

where $r = r_i = \frac{11}{16}$ in., $J = \frac{1}{2}\pi r_i^4 = 0.350$ in^4, and $T = T'$ of part (a) $= 0.230W$.

$$\text{Maximum shear stress, } \tau(\max) = \sqrt{(s_c/2)^2 + s_s^2}$$

$$4000 = \sqrt{(0.674W/2)^2 + (0.452W)^2} \quad \text{or} \quad W = 7100 \text{ lb}$$

7. A hand-operated valve grinding device is to be operated by forcing a nut downward along the stem which is provided with helical grooves square in axial section. The *overhauling* action thereby causes the stem to rotate and turn the valve in its seat either by means of a screwdriver tip or by means of a suction cup, depending on the type of valve. Assume the following data in addition to that shown on the sketch, Fig. 12-9 below.

Coefficient of friction f between nut and stem = 0.10.
Coefficient of friction f_c between valve and seat = 0.35.
Mean friction radius between valve and seat = 1.0 in.

Determine the minimum helix angle α which could be used in the device under the conditions assumed.

Solution:

The helix angle must be at least large enough to guarantee *overhauling* action. This would be the value which makes $T = 0$ in the equation

Bolt Loading

D JOINTS LOADED IN TENSION are frequently encountered in the design of fasteners. The bolt is subjected to an initial load in tension, W_1, often ed by the application of an external load, W_2, as shown in Fig. 13-1. The resultant load W on lt is determined by

$$W = W_1 + W_2\left(\frac{m_1 + m_2 + m_3 + \dots + m_n}{b + m_1 + m_2 + m_3 + \dots + m_n}\right)$$

$$W = W_1 + W_2\left(\frac{m}{m + b}\right)$$

ere W_1 = initial load on bolt due to tightening, lb

W_2 = external load, lb

W = resultant load on bolt due to W_1 and W_2, lb

m_1, m_2 and m_3 are defined as the deflection in in. per lb of load for the bolted members, m_1, m_2 and m_3. These symbols refer to all parts in the bolted assembly, including the gasket.

m = the sum of m_1, m_2, etc.

b is defined as the deflection in in. per lb of load for the bolt.

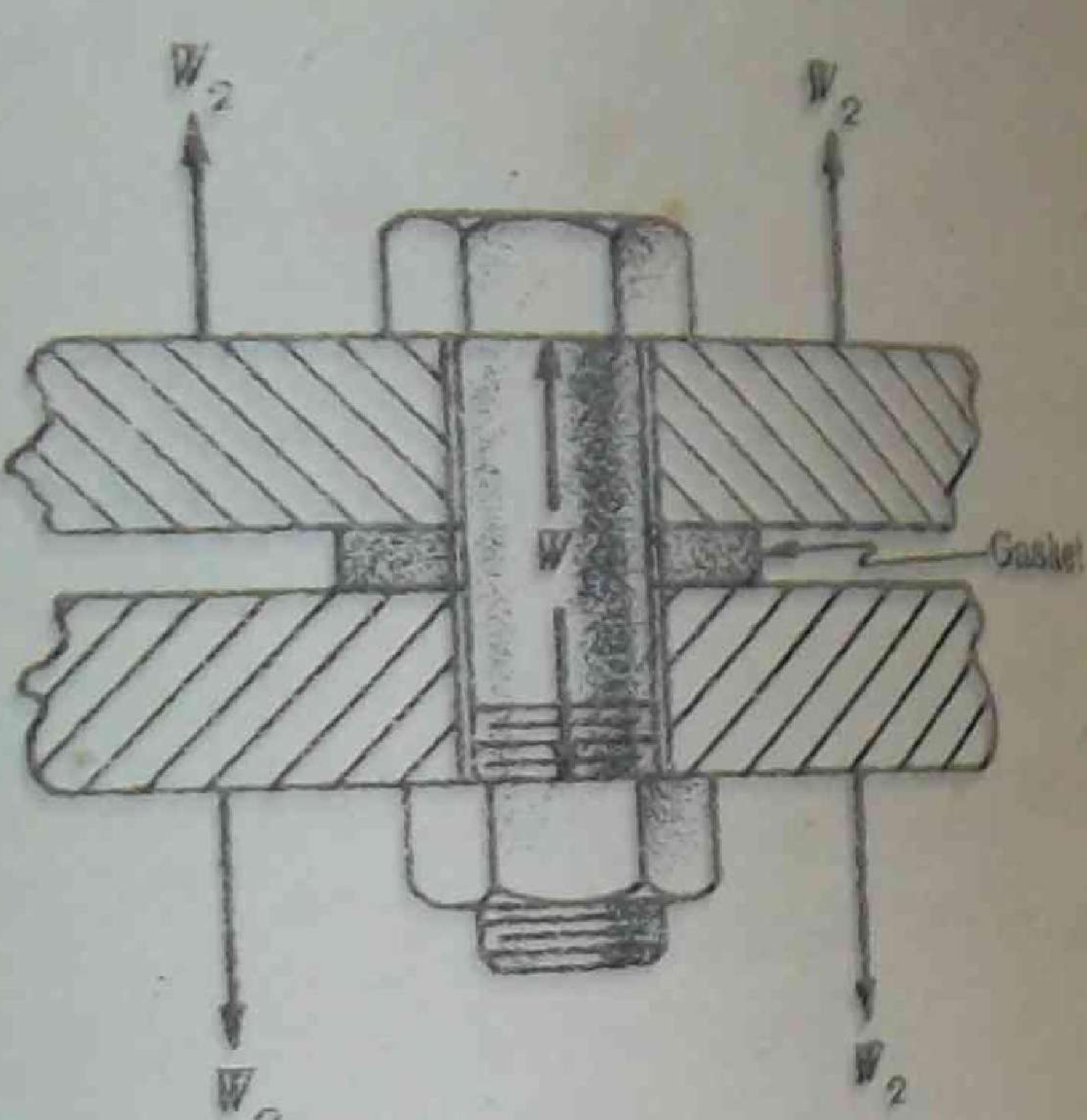


Fig. 13-1

SEPARATION OF THE BOLTED JOINT will occur if $W_2 = W$. The equation given above is that of a straight line having slope $m/(m+b)$ and ordinate intercept W_1. The plot of this line, as shown in Fig. 13-2, provides a quick method for determining when separation of the members will occur. Line AC is the extreme situation of zero slope which occurs when the members have practically no deflection per lb of load as compared to the bolt, i.e. $m/b = 0$. Line AB represents the extreme situation when the bolt has practically no deflection per lb of load as compared to the members, i.e. $b/m = 0$. The actual situation occurs between these two extremes, as indicated by lines AD and DE.

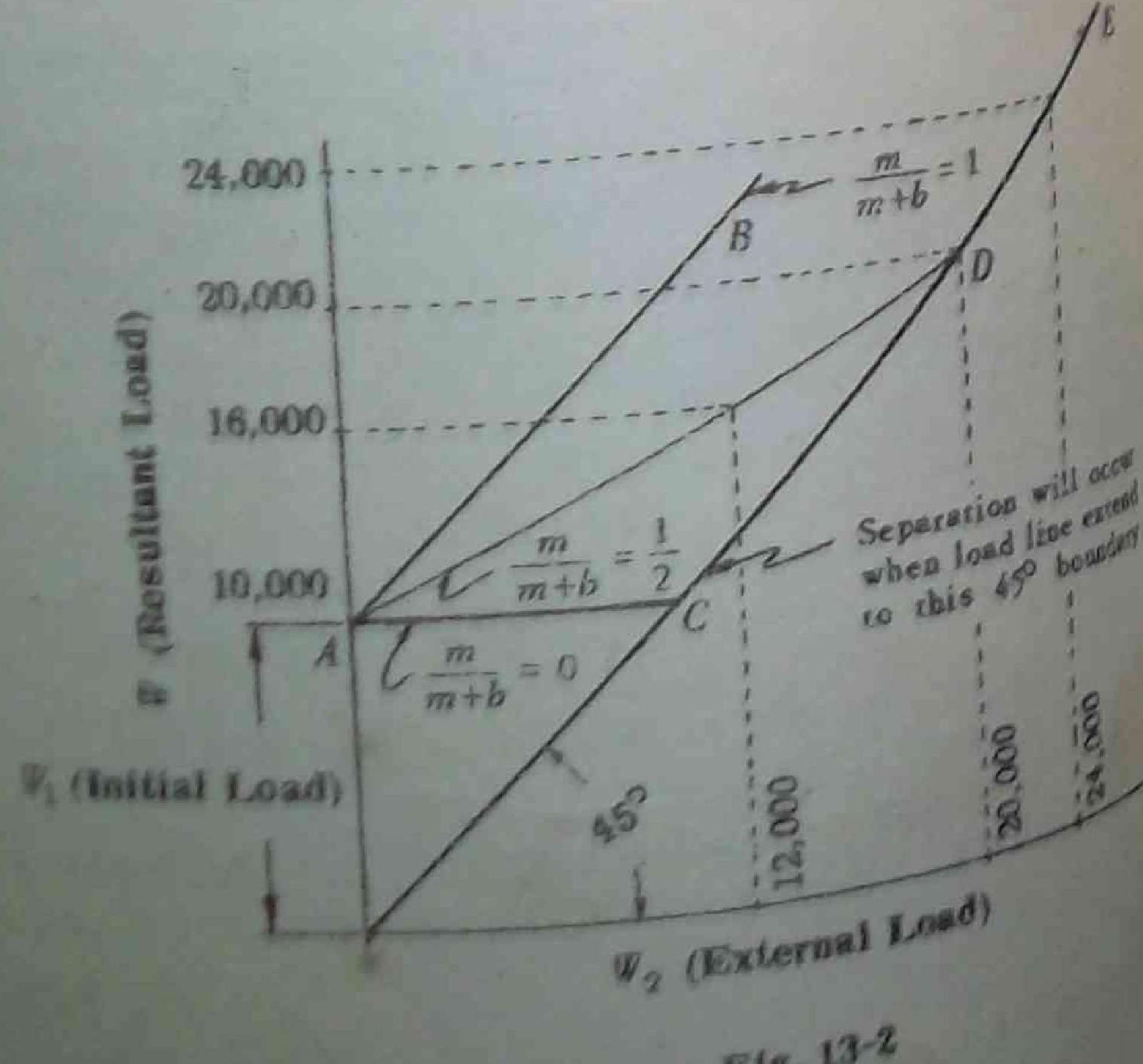


Fig. 13-2

THE INITIAL TENSION IN THE BOLT due to tightening may result from the use of a torque wrench or may depend upon the judgment of an experienced mechanic.

Experimental results indicate that the initial bolt load W_i in a bolt tightened by an experienced mechanic may be estimated by

$$W_i = Kd$$

where d = nominal bolt diameter and K ranges from about 10,000 to 16,000.

If a torque wrench is used to tighten the bolt, the value of the initial bolt load W_i may be approximated by

$$W_i = T/0.2d$$

where T = applied tightening torque and d = nominal bolt diameter. The above equation is obtained by using the screw thread torque equation (Chapter 12), neglecting the helix angle, assuming a coefficient of friction 0.15 for both threads and nut, and assuming the friction collar radius of the nut is approximately 2/3 of the pitch diameter of the bolt. For well lubricated threads, the initial bolt load might be as much as double that indicated by the above expression. The use of the screw thread torque equation is preferred to the above approximation.

The initial bolt load may also be calculated from the theoretical expression

$$T = \text{thread torque} + \text{collar torque} = W_i r_m \left(\frac{\tan \alpha + f/\cos \theta_n}{1 - f \tan \alpha / \cos \theta_n} \right) + W_i f_c r_c$$

where
- r_m = mean radius of thread, in.
- f = coefficient of thread friction
- f_c = coefficient of collar friction
- α = helix angle
- θ_n = one-half of thread angle
- r_c = collar friction radius, in.

STRESSES INDUCED IN TENSION BOLTS are the result of torsional shear combined with resultant axial bolt load. The maximum shear stress in a tension bolt may be calculated by

$$\tau(\max) = \sqrt{(W/2A_r)^2 + (16T_f/\pi d_r^3)^2}$$

where
- $\tau(\max)$ = maximum shear stress in the body of the screw, psi
- A_r = root area, in^2
- T_f = thread torque, in-lb
- d_r = root diameter of thread, in.
- W = resultant axial bolt load, lb

For a less conservative design A_r and d_r may be replaced by A_s and d_s based on the stress area A_s which is a mean of the average pitch diameter area and the average minor diameter area for Class 3A tolerances, and $d_s = \sqrt{4A_s/\pi}$

In general for static loading, the maximum shear stress induced in the bolt should not exceed about 3/4 of the shear yield strength of the material; however, there are times, especially in the case of small bolts ($\frac{1}{2}$ in. and less), where the yield point is exceeded. For variable loading the bolt should be designed for endurance. It should also be noted that bolts usually lose their initial torsional stress when they are subjected to dynamic loading. This is due to the fact that the bolt head and/or nut will slip back if there is insufficient frictional collar resistance.

IMPACT STRESSES result when bolts are subjected to suddenly applied or impact loads. The bolt has to absorb the energy of impact.

$$U = \tfrac{1}{2}F\delta$$

where F = force caused by impact, lb; δ = deformation caused by impact, in.; U = energy of impact, in-lb.

REQUIRED HEIGHT OF THE NUT may be estimated assuming that each turn of the thread supports an equal part of the resultant load W, as shown in Fig. 13-3.

The strength of the bolt in tension should equal the strength of the threads in shear. For the bolt in tension, $W = \frac{1}{4}\pi d_r^2 s_t$; for the threads in shear, $W = \pi d_r h s_s$; for ductile materials, $s_s = \frac{1}{2}s_t$. Then

$$\frac{1}{4}\pi d_r^2 = \frac{1}{2}\pi d_r h \qquad \text{or} \qquad h = \frac{1}{2}d_r$$

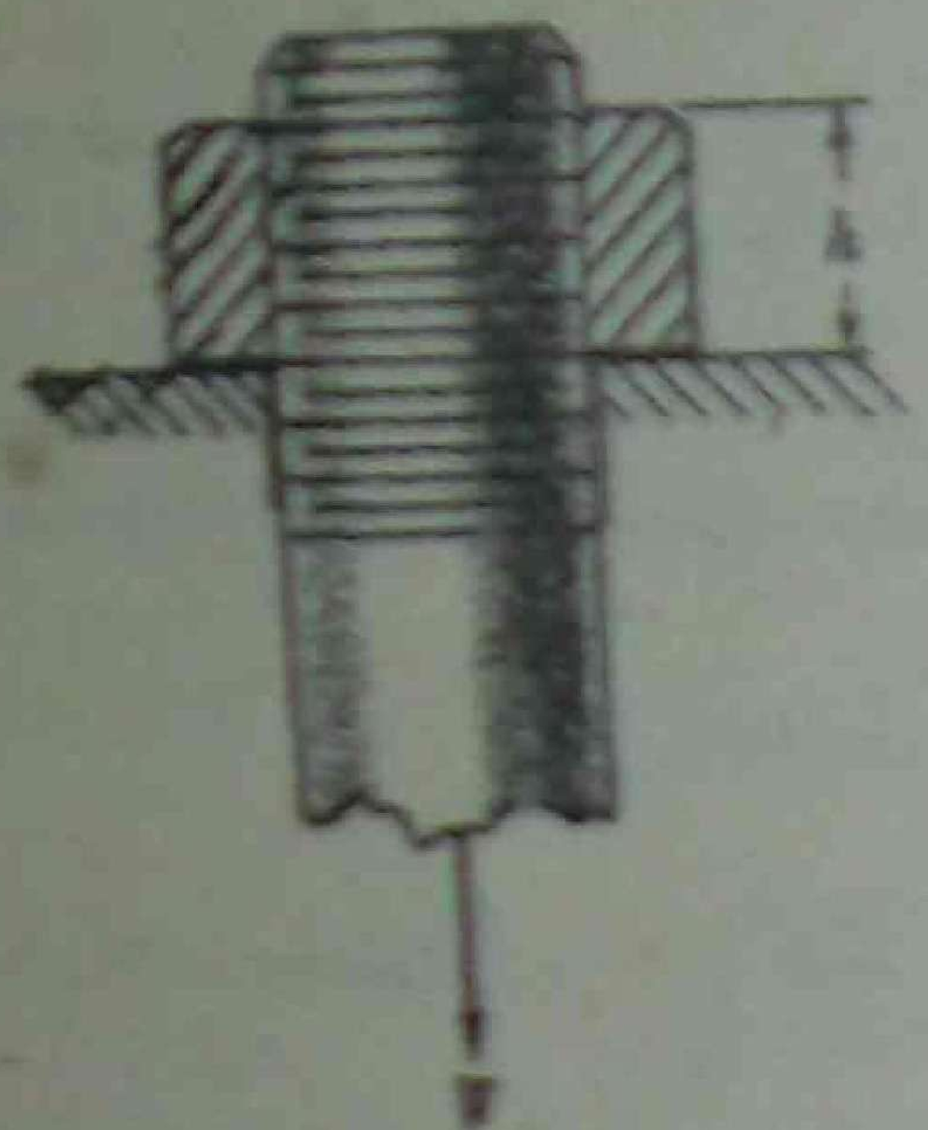


Fig. 13-3

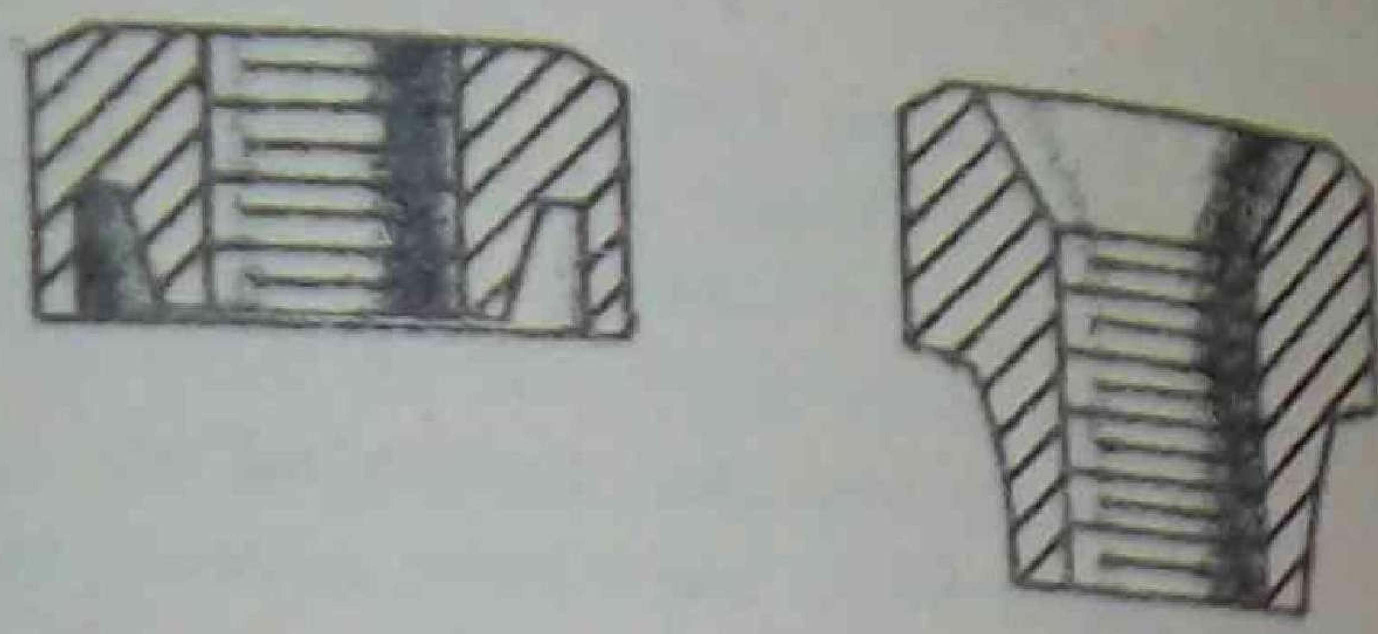


Fig. 13-4

The assumption that each thread takes its share of the load is not true. Since the nut is under compression and the bolt is under tension, the load will shift toward the bottom threads of the nut. For heavily loaded bolts, special nuts are sometimes used in order to obtain better load distribution as shown in Fig. 13-4 above.

THE FATIGUE STRENGTH OF A BOLT depends upon the maximum and minimum loads to which it is subjected. When the external load W_2 is fluctuating, the initial bolt load W_1 should be sufficient to prevent separation with a reasonable factor of safety. Separation will be pending when the external load W_2 is equal to the resultant bolt load W. Then

$$W_2 = W_1 + W_2\left(\frac{m}{m+b}\right)$$

or W_1 must be $\geq W_2\left(1 - \frac{m}{m+b}\right)$ to prevent separation. When no separation occurs, the load may vary between W_1 and $W_1 + W_2\left(\frac{m}{m+b}\right)$.

STRESS CONCENTRATION AT THE ROOT of a standard coarse thread is high. Photoelastic tests indicate static stress concentration factors as high as 5.62. This may not be too serious for bolts made of ductile material when subject to static loads. However, the stress concentration factor has been found to lower the endurance limit of standard coarse threads by factors ranging from 2 to 4. Therefore the fluctuating stress in a threaded bolt must be multiplied by a suitable stress concentration factor.

SOLVED PROBLEMS

1. Derive the equation for the resultant bolt load W in terms of the initial load W_1 and the applied external load W_2.

Solution:

Consider two members bolted together. Fig. 13-5 below shows the members and the bolt as free bodies due to the initial tightening load W_1 only. Fig. 13-6 shows the members and the bolt as free bodies after an external load W_2 has been applied. Note that the change in length of the bolt is equal to the change in length of the bolted members if there is no separation of the parts. Then

$$(\Delta L)_b = (\Delta L)_1 + (\Delta L)_2$$

and
$$(W - W_1)b = [W_1 - (W - W_2)]m_1 + [W_1 - (W - W_2)]m_2$$

or
$$W = W_1 + W_2\left(\frac{m_1 + m_2}{b + m_1 + m_2}\right) = W_1 + W_2\left(\frac{m}{m + b}\right)$$

where the symbols are the same as previously given. (Note: external load is assumed to be applied to the bolt.)

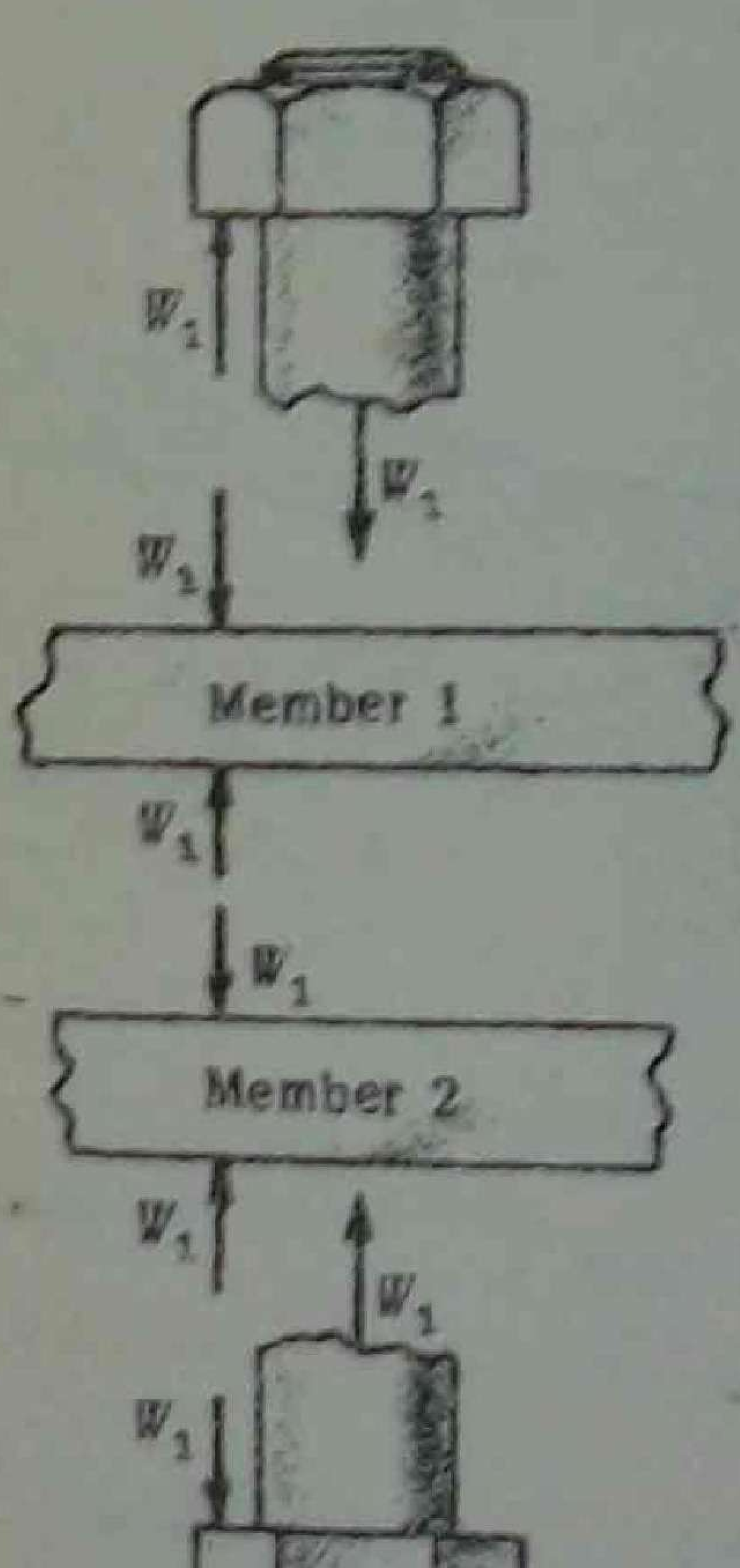


Fig. 13-5

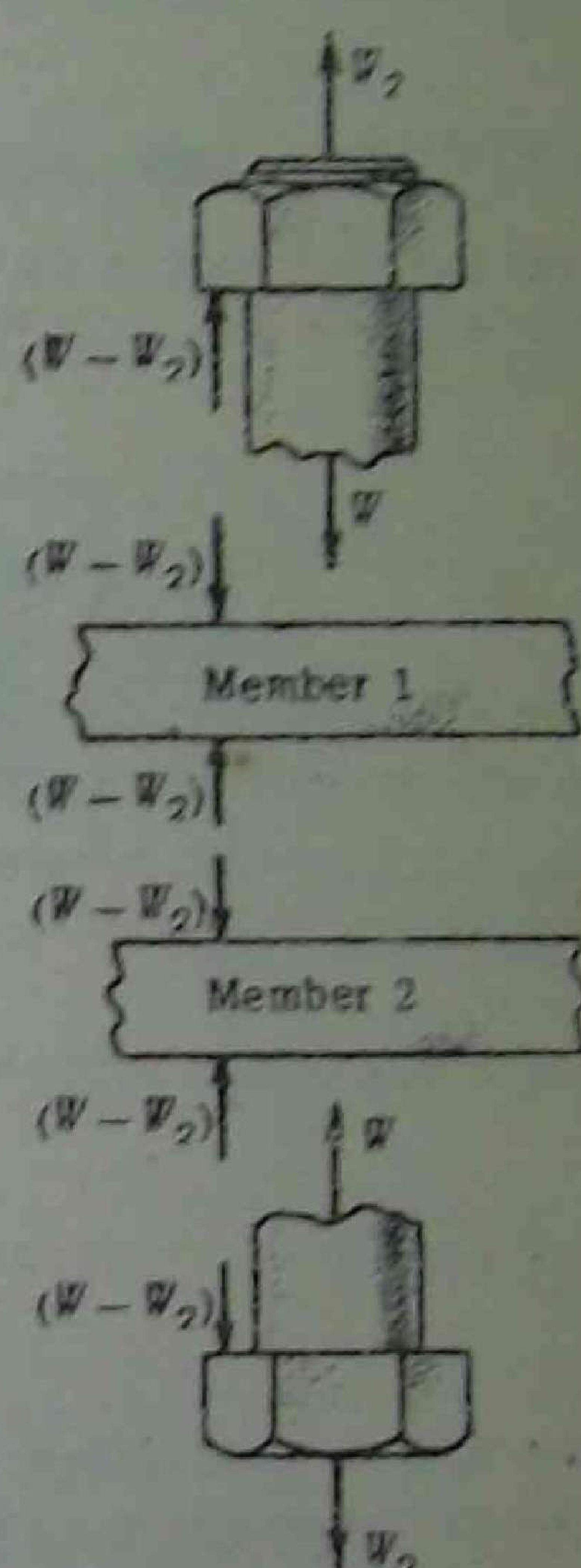


Fig. 13-6

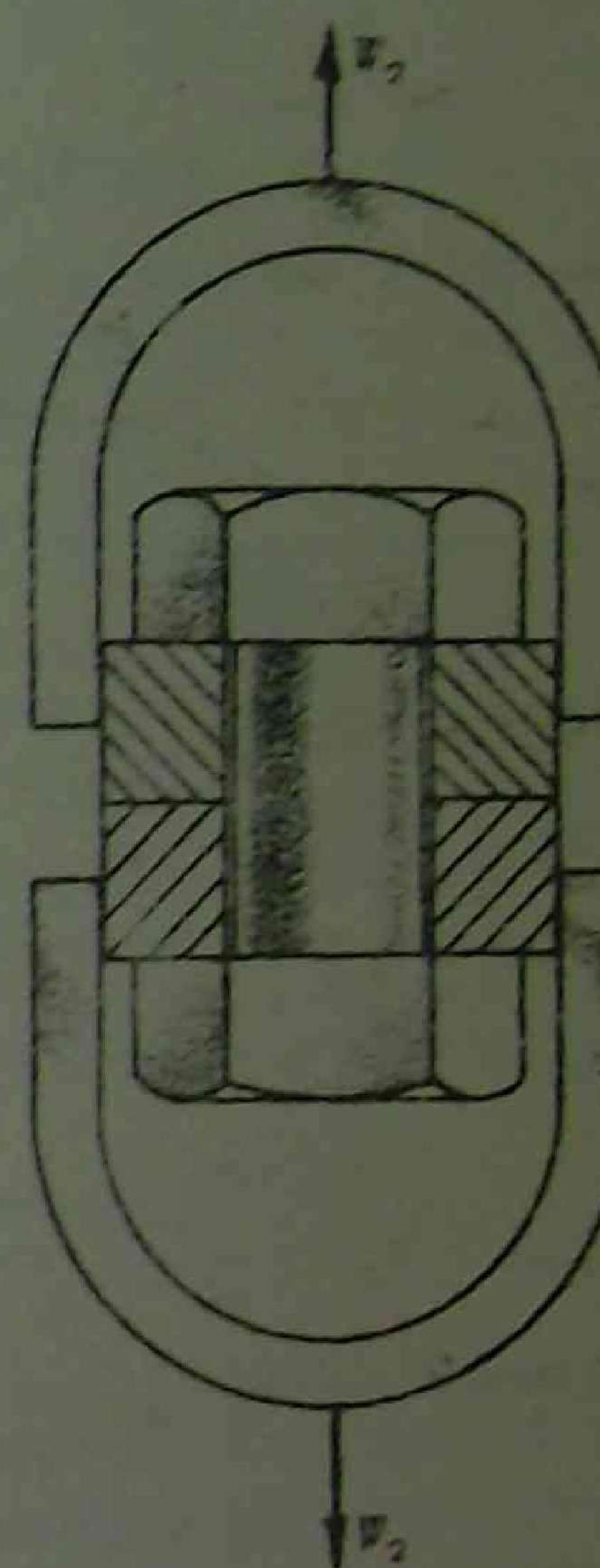


Fig. 13-7

2. A bolt is used to fasten two members together as shown in Fig. 13-7 above. The members and the bolt are of the same material and have the same cross section area. Determine what external load W_2 will cause separation of the members to occur if the initial tightening load W_1 is 5000 lb.

Solution:

Since the connected members and bolt are of the same material and have the same cross section area, they have the same deflection per lb of load. Thus $m = b$ and $W = W_1 + W_2\left(\frac{m}{m+b}\right) = W_1 + \frac{1}{2}W_2$.

Separation will occur when $W = W_2$; then $W_2 = 5000 + \frac{1}{2}W_2$, or $W_2 = 10{,}000$ lb before separation occurs.

3. Several members are bolted together in such a manner that the deflection per lb of load for the bolted members is the same as for the bolt, i.e. $m = b$ or $m/(m+b) = \frac{1}{2}$. Determine the following graphically using Fig. 13-2.

(a) If the initial tightening load on the bolt is 10,000 lb, what axial external load has to be applied to the bolt to cause separation of the bolted members?

(b) What is the resultant bolt load for an external load of 12,000 lb?

(c) What is the resultant bolt load for an external load of 24,000 lb?

Solution: Refer to Fig. 13-2.

(a) Separation occurs at point D for which an external load $W_2 = 20{,}000$ lb will just cause separation.

(b) For an external load $W_2 = 12{,}000$ lb, the resultant bolt load $W = 16{,}000$ lb.

(c) For an external load $W_2 = 24{,}000$ lb, the resultant bolt load $W = 24{,}000$ lb (separation has occured).

4. The bolted assembly shown in Fig. 13-8 below has been preloaded by tightening the nut so that the bolt has an initial load of 1200 lb. If the ratio of the deflection per lb of load for the members to the deflection per lb of load for the bolt is 1/3, what is the magnitude of the bolt load when an external load $W_2 = 2000$ lb is applied as shown?

Solution:

Since $m/b = 1/3$, $\quad W = W_1 + W_2(\frac{m}{m+b}) = W_1 + \frac{1}{4}W_2 = 1200 + \frac{1}{4}(2000) = 1700$ lb.

Since $W_2 > W$, the members have separated and the final load on the bolt will be 2000 lb.

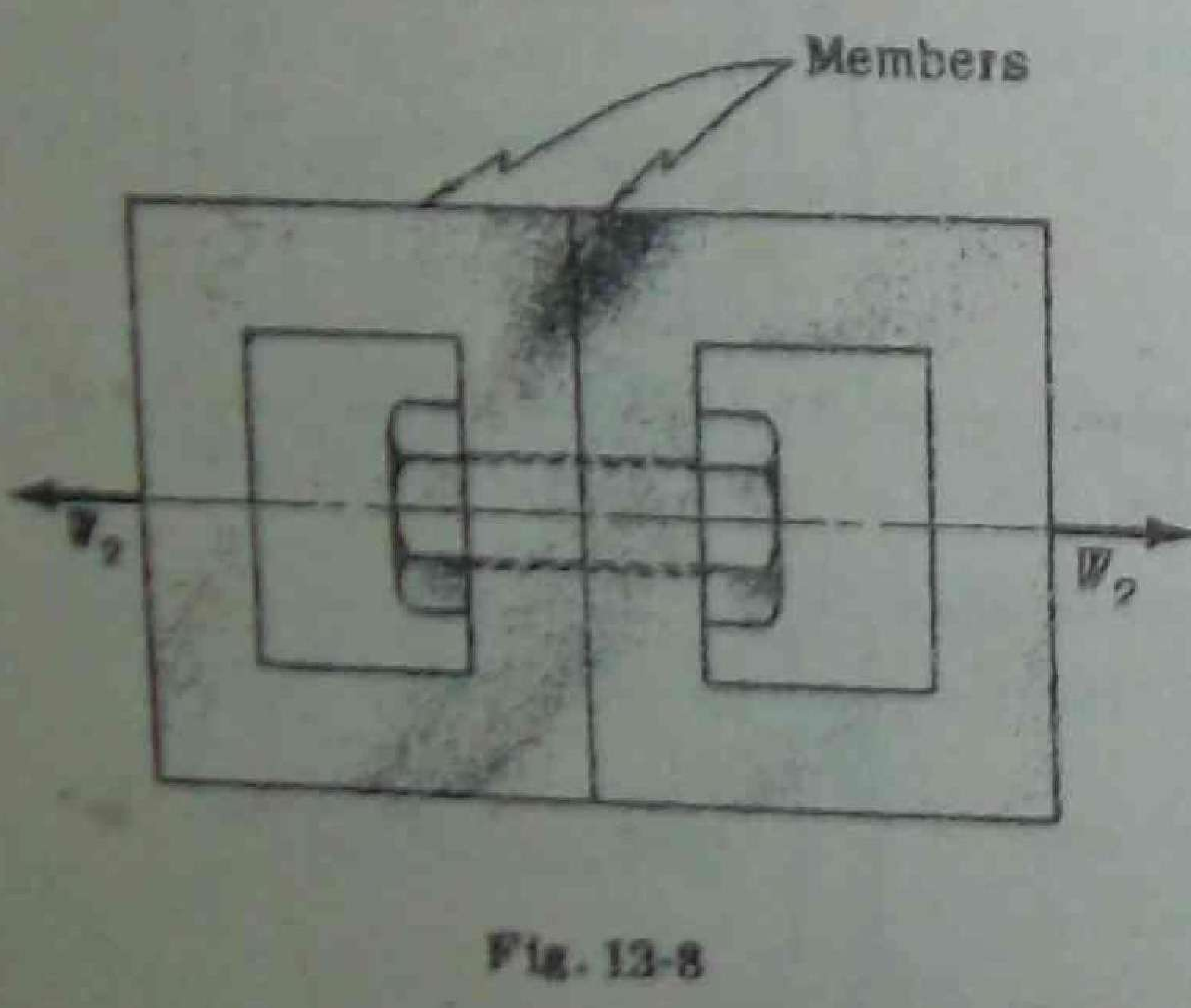


Fig. 13-8

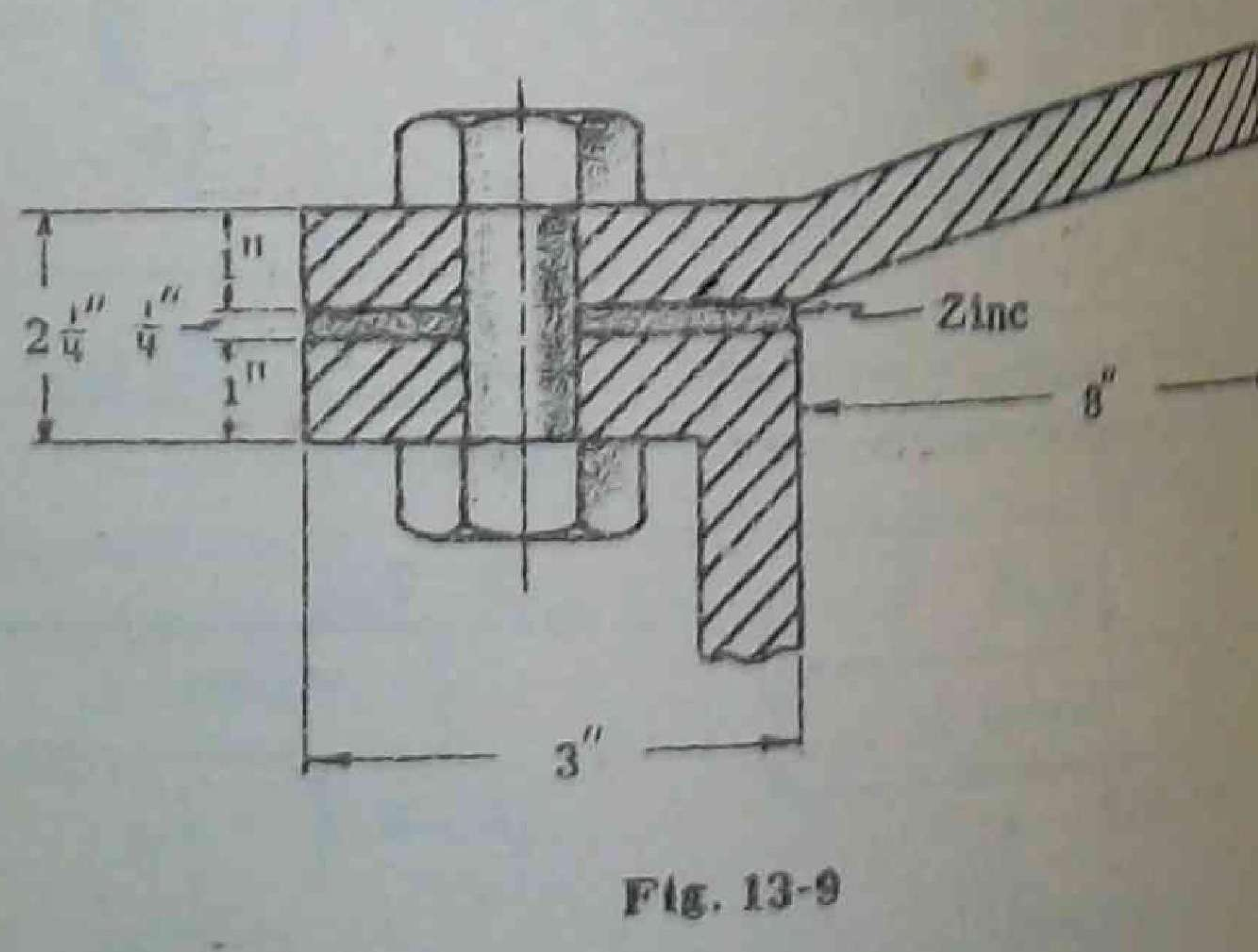


Fig. 13-9

5. The lid of a cast iron pressure vessel shown in Fig. 13-9 above is held in place by ten $\frac{1}{2}$ in. steel bolts having an initial tightening load of 5000 lb, when the vessel is at 70°F and the initial pressure is atmospheric. Determine the load in each bolt (a) if the pressure is increased to 200 psi, (b) if the vessel is heated to 250°F with atmospheric internal pressure, (c) if the vessel is heated to 250°F with an internal pressure of 200 psi.

E for steel $= 30\times10^6$ psi, for cast iron $= 12\times10^6$ psi, for zinc $= 12\times10^6$ psi.

Linear expansion coefficient for steel $= 6.6\times10^{-6}/°F$, for cast iron $= 5.6\times10^{-6}/°F$, for zinc $= 17.8\times10^{-6}/°F$.

Solution:

(a) Cross section area of the flange and zinc gasket per bolt is

$$A = \frac{1}{10}(\pi D_m t) = \frac{1}{10}\pi(\frac{16+22}{2})(3) = 17.9 \text{ in}^2$$

Cross section area of $\frac{1}{2}$ in bolt $= 0.196$ in^2

Hence the net area for zinc and cast iron flange $= 17.9 - 0.196 = 17.7$ in^2/bolt.

From $s = \frac{\Delta L}{L}E = \frac{P}{A}$, we have $\frac{\Delta L}{P} = \frac{L}{AE}$ inches deflection per lb of load.

$$(L/AE)_{zinc} = \frac{0.25}{(17.7)(12)(10^6)} = \frac{1}{(850)(10^6)} \text{ in/lb of load}$$

$$(L/AE)_{CI} = \frac{2.0}{(17.7)(12)(10^6)} = \frac{1}{(106)(10^6)} \text{ in/lb of load}$$

$$(L/AE)_{bolt} = \frac{2.25}{(0.196)(30)(10^6)} = \frac{1}{(2.62)(10^6)} \text{ in/lb of load}$$

$$m = \frac{1}{(350)(10^6)} + \frac{1}{(106)(10^6)} \text{ in/lb of load}, \quad b = \frac{1}{(2.62)(10^6)} \text{ in/lb of load}, \quad \frac{m}{m+b} = \frac{1}{37}$$

$$W_2 = \frac{\pi(16^2)}{4}\left(\frac{200}{10}\right) = 4020 \text{ lb per bolt, and } W = 5000 + \frac{4020}{37} = 5110 \text{ lb}$$

(*b*)

$$(\Delta L)_{zinc} \text{ due to temp. change} = \frac{(0.25)(180)(17.8)}{10^6} = \frac{(180)(4.45)}{10^6} \text{ in}$$

$$(\Delta L)_{CI} \text{ due to temp. change} = \frac{(2)(180)(5.6)}{10^6} = \frac{(180)(11.2)}{10^6} \text{ in}$$

$$(\Delta L)_{bolt} \text{ due to temp. change} = \frac{(2.25)(180)(6.6)}{10^6} = \frac{(180)(14.9)}{10^6} \text{ in}$$

$$(\Delta L)_{zinc} + (\Delta L)_{CI} = \frac{180}{10^6}(4.45 + 11.2) = \frac{180}{10^6}(15.65) \text{ in}$$

$(\Delta L)_{bolt} = (\Delta L)_{members}$. $(\Delta L)_{bolt}$ will increase due to both an increase in temperature and an increase in load. Let W_1' be the new load on the bolt, remembering that $W_1 = 5000$ lb.

$$(\Delta L)_{bolt} = \frac{(W_1' - 5000)(2.25)}{(0.196)(30)(10^6)} + \frac{(180)(14.9)}{10^6}$$

$(\Delta L)_{members}$ will tend to decrease due to an increase in load, but will tend to increase due to the temperature rise; thus

$$(\Delta L)_{members} = -\frac{(W_1' - 5000)(2.25)}{(17.7)(12)(10^6)} + \frac{(180)(15.65)}{10^6}$$

Equating $(\Delta L)_{bolt} = (\Delta L)_{members}$, we obtain $W_1' = 5360$ lb. This is the new initial bolt load.

(*c*) After the external pressure has been applied the resultant bolt load will be $W = 5360 + 4020/37 = 5469$ lb.

6. A $\frac{1}{2}$ in. 12 UNC by 10 in. long steel bolt is subjected to an impact load. The kinetic energy to be absorbed is 35 in-lb. (*a*) Determine the stress in the shank of the bolt if there is no threaded portion between the nut and the bolt head. (*b*) Find the stress in the shank if the area of the shank is reduced to that of the root area of the threads.

Solution:

(*a*) The energy of impact will be absorbed by elongation of the $\frac{1}{2}$ in. shank which has an area of 0.196 in^2.

$$U = \frac{F}{2}\delta = \frac{F}{2}\left(\frac{FL}{AE}\right) = \frac{F^2L}{2AE} \quad \text{or} \quad F = \sqrt{\frac{2AEU}{L}} = \sqrt{\frac{(2)(0.196)(30)(10^6)(35)}{10}} = 6400 \text{ lb (impact load)}$$

The root area $A_r = 0.1257$ in^2. The stress based on the root area is $s = 6400/0.1257 = 50,900$ psi. This value has neglected stress concentration.

Chapter 14

Clutches

A CLUTCH is a friction device which permits the connection and disconnection of shafts. The designs of clutches and brakes are comparable in many respects. This is well illustrated by a multiple disk clutch which is used also as a brake. One problem of design much more evident in brake design compared to clutch design is that of heat generation and dissipation. Friction clutches generate heat as a result of relative motion of the parts, but the amount of sliding is not ordinarily as great as in a brake. It is quite customary in the analysis of a clutch to picture that the parts are in impending motion with respect to each other, although one must not lose sight of the fact that transmission of power through friction usually involves some slip. For this reason, when it is necessary to have positive power transmission one must resort to a positive device, as a jaw type of clutch.

PLATE OR DISK CLUTCHES

A MULTIPLE DISK clutch is shown in Fig. 14-1. The plates shown as A are usually steel and are set on splines on shaft C to permit axial motion (except for the last disk). The plates shown as B are usually bronze and are set in splines of member D.

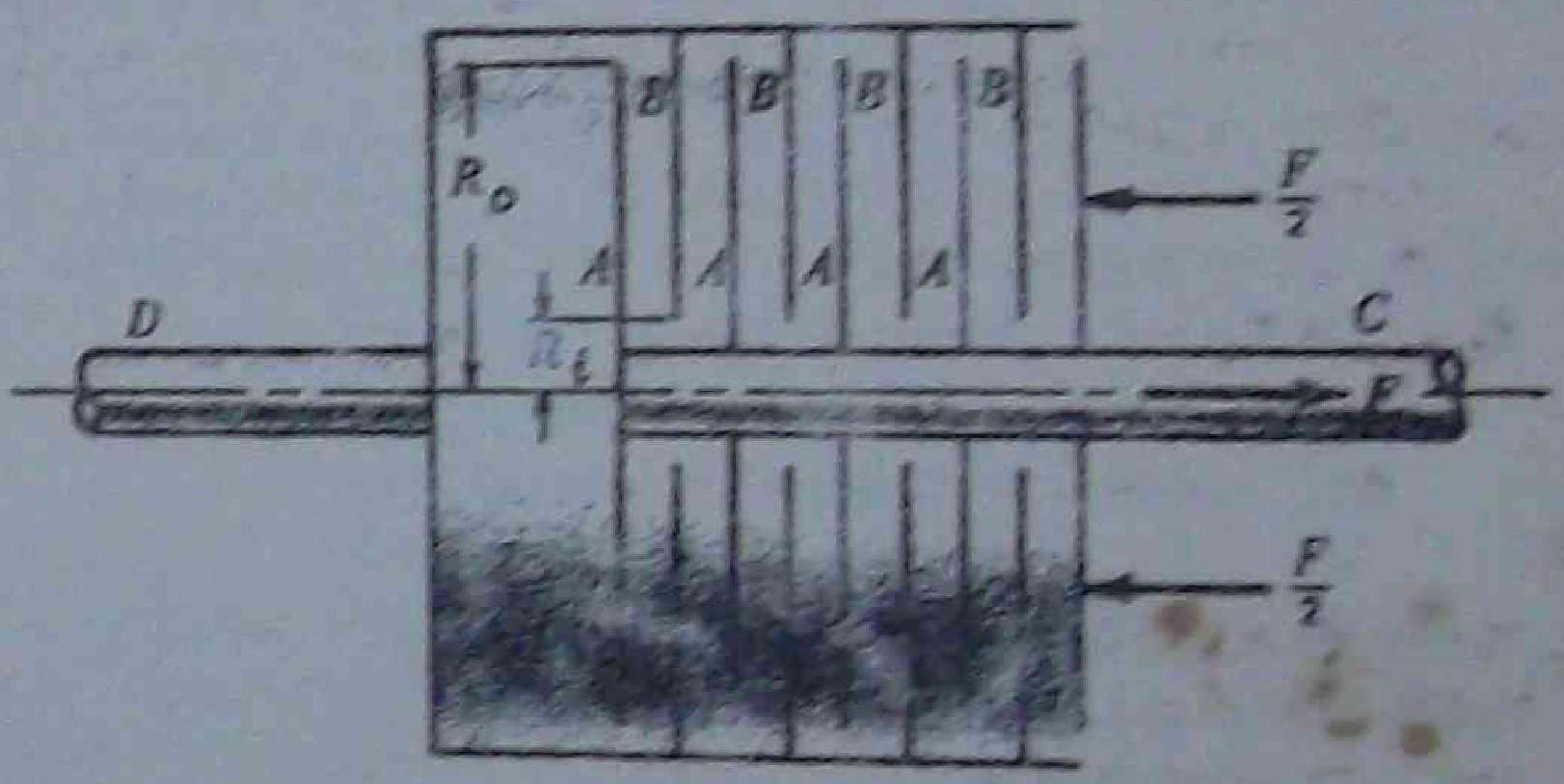


Fig. 14-1

The number of pairs of surfaces transmitting power is one less than the sum of the steel and bronze disks, and is also an even number if the design is such that no thrust bearings are needed.

$$n = n_{steel} + n_{bronze} - 1$$

For the system shown, $n = 5 + 4 - 1 = 8$ pairs of surfaces in contact.

The torque capacity is given by

$$T = FfR_f n$$

where
T = torque capacity, in-lb
F = axial force, lb
f = coefficient of friction
R_f = friction radius $= \frac{2}{3}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)$ if the contact pressure is assumed uniform
$= \frac{R_o + R_i}{2}$ if wear is assumed uniform
R_o = outside radius of contact of surfaces, inches
R_i = inside radius of contact of surfaces, inches
n = number of pair of surfaces in contact.

The axial force ... is

$$F = p\pi(R_o^2 - R_i^2)$$

where p is the average pressure.

The horsepower capacity is

$$\text{Hp} = TN/63{,}000$$

where T = shaft torque, in-lb; and N = speed of rotation, rpm.

For uniform wear, the pressure variation is given by

$$p = \frac{C}{r} = \frac{F}{2\pi(R_o - R_i)r}$$

where C is a constant and r is radius to differential element shown in Fig. 14-3, on page 168.

CONE CLUTCHES

A CONE CLUTCH achieves its effectiveness by the wedging action of the cone part in the cup part.

(a) The torque capacity of a cone clutch with the parts engaged based on uniform pressure is

$$T = \frac{Ff}{\sin\alpha}\left[\frac{2}{3}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)\right]$$

The torque capacity can also be written in an alternate form as

$$T = Ff\left(\frac{R_o^3 - R_i^3}{3R_m b \sin^2\alpha}\right)$$

where T = torque, in-lb
F = axial force, lb
f = coefficient of friction
R_o = outside radius of contact, in.
R_i = inside radius of contact, in.
R_m = mean radius = $\frac{1}{2}(R_o + R_i)$, in.
b = face width, in.
α = pitch cone angle.

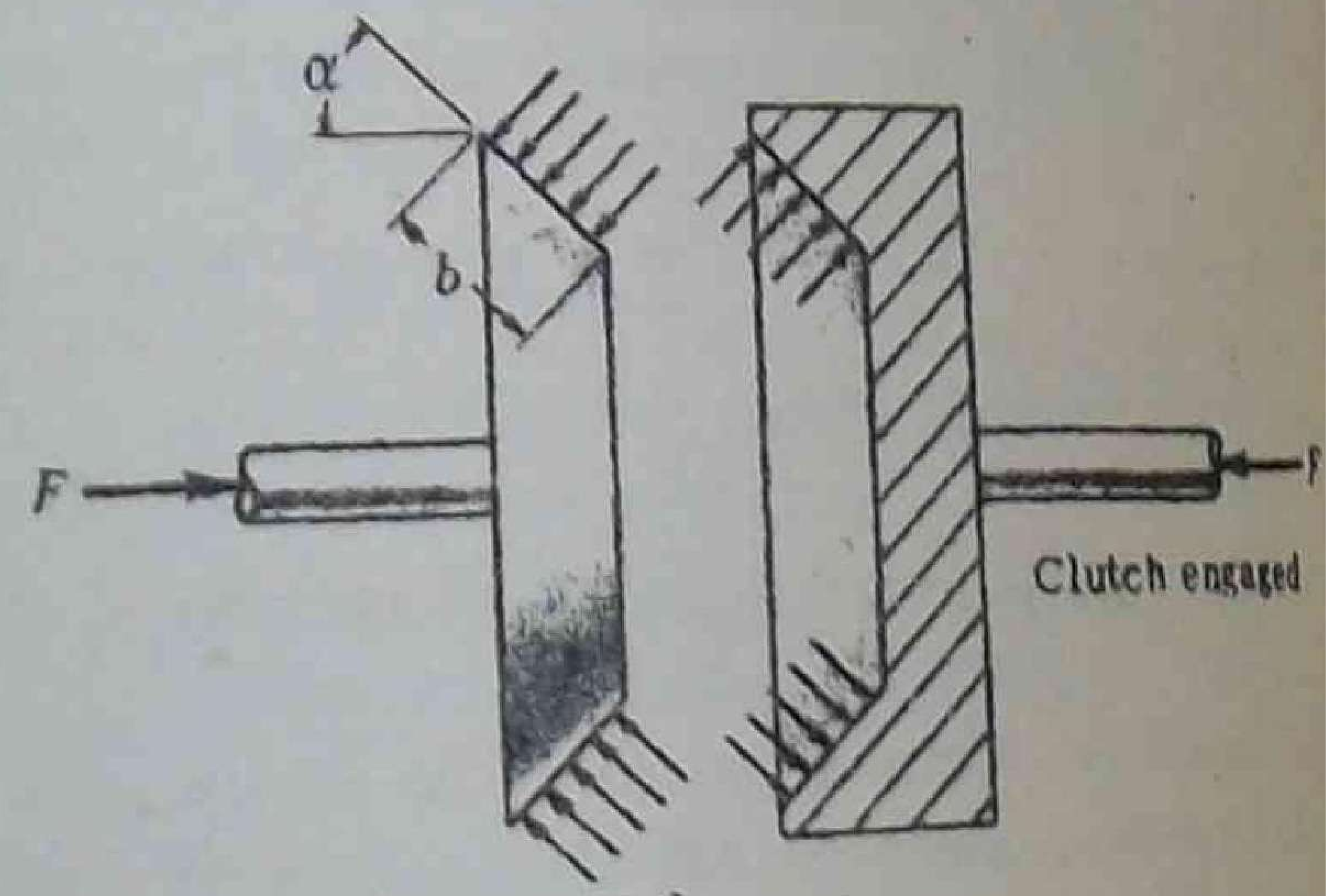


Fig. 14-2

or $T = F_n f\left[\frac{2}{3}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)\right]$ where $F_n = p(2\pi R_m)(b)$.

(b) The torque capacity of a cone clutch, based on uniform wear is given by

$$T = \frac{FfR_m}{\sin\alpha} \quad \text{or} \quad T = F_n f R_m$$

The pressure variation, where uniform wear is assumed, is

$$p = \frac{F}{2\pi(R_o - R_i)r}$$

See Fig. 14-8

The maximum pressure occurs at the smallest radius: $p_{max} = \frac{F}{2\pi(R_o - R_i)R_i}$.

The minimum pressure occurs at the largest radius: $p_{min} = \frac{F}{2\pi(R_o - R_i)R_o}$.

The average pressure $= \frac{F}{\pi(R_o^2 - R_i^2)}$

6. It is usual practice to design a hub such that its outside diameter is about twice the bore diameter. It is also known that a selective assembly should be used when shrinking a hub on a shaft. The purpose of this problem is to determine how small and how large stresses can be with a shrink fit (class 8 fit) if a selective assembly is **not** used. Determine for a 1" diameter solid shaft, the actual maximum and minimum **tangential** stresses that will result if the maximum and minimum interferences should be used for a hub with a 2" outside diameter. The shaft and hub are both made of steel. Poisson's ratio may be taken as equal to 0.3.

Solution:

$d_i = 0, \quad d_c = 1'', \quad d_o = 2''$

First determine the radial pressure on the contact surface, p_c. Since both hub and shaft are made of the same material,

$$p_c = \frac{\delta E(d_c^2 - d_i^2)(d_o^2 - d_c^2)}{2d_c^3(d_o^2 - d_i^2)} = \frac{\delta(30)(10^6)(1^2 - 0)(2^2 - 1^2)}{(2)(1^3)(2^2 - 0)} = \delta(11.25)(10^6)$$

Then, using Lamé's equation, determine the tangential stress at the contact surface of the outer member.

$$s_{tco} = p_c \frac{d_o^2 + d_c^2}{d_o^2 - d_c^2} = \delta(11.25)(10^6)\frac{2^2 + 1^2}{2^2 - 1^2} = \delta(18.75)(10^6)$$

For a class 8 fit, hole dimension may vary from 1.0000" to 1.0006" and shaft dimension may vary from 1.0010" to 1.0016"; then $\delta(max) = 0.0016''$, $\delta(min) = 0.0004''$, and

$$s_{tco}(max) = (0.0016)(18.75)(10^6) = 30{,}000 \text{ psi}$$
$$s_{tco}(min) = (0.0004)(18.75)(10^6) = 7{,}500 \text{ psi}$$

7. A 6 in. diameter steel shaft is to have a press fit with a 12 in. o.d. by 10 in. long hub of cast iron. The maximum tangential stress is to be 5000 psi. $E = 30 \times 10^6$ psi for steel and 15×10^6 psi for cast iron; $\mu = 0.3$ assumed for both steel and cast iron; $f = 0.12$.
(*a*) Determine the maximum diametral interference.
(*b*) What axial force F_a will be required to press the hub on the shaft?
(*c*) What torque may be transmitted with this fit?

Solution:

(*a*) The tangential maximum stress occurs on the surface d_c for the outer member

$$s_{tco} = p_c\left(\frac{d_o^2 - d_c^2}{d_o^2 + d_c^2}\right), \quad 5000 = p_c\left(\frac{12^2 + 6^2}{12^2 - 6^2}\right), \quad p_c = 3000 \text{ psi}$$

Using

$$p_c = \frac{\delta}{d_c\left[\dfrac{d_c^2 + d_i^2}{E_i(d_c^2 - d_i^2)} + \dfrac{d_o^2 + d_c^2}{E_o(d_o^2 - d_c^2)} - \dfrac{\mu_i}{E_i} + \dfrac{\mu_o}{E_o}\right]}$$

$$3000 = \frac{\delta}{6\left[\dfrac{6^2 + 0}{(30)(10^6)(6^2 - 0)} + \dfrac{12^2 + 6^2}{(15)(10^6)(12^2 - 6^2)} - \dfrac{0.3}{(30)(10^6)} + \dfrac{0.3}{(15)(10^6)}\right]}$$

from which $\delta = 0.00278$ in. (maximum permissible diametral interference)

(*b*) $F_a = f\pi d L p_c = 0.12\pi(6)(10)(3000) = 67{,}800$ lb

(*c*) $T = f p_c \pi d^2 L/2 = F_a(d/2) = 67{,}800(6/2) = 203{,}400$ in-lb

ENGAGING CONE CLUTCHES. A problem encountered with cone clutches not encountered with multi-disk clutches is the possibility of a larger force to engage a clutch than that required during operation when the cup and cone are rotating at the same speed. The analysis is complicated by the fact that the direction of the frictional forces depends upon the manner of engagement, that is, the ratio of the relative rotary motion to the relative axial motion of the cup and cone. A conservative procedure is to assume that no relative rotary motion occurs during engagement, for which the maximum axial force F_e necessary to engage the cup and cone is

$$F_e = F_n(\sin\alpha + f\cos\alpha)$$

This force is the maximum required to obtain the desired normal force F_n which in turn develops the frictional force to give the desired frictional torque.

AXIAL FORCE TO HOLD CUP AND CONE IN ENGAGEMENT. The force to hold the cup and cone in engagement, with friction taken into account, will vary between

$$F = F_n \sin\alpha \qquad \text{and} \qquad F = F_n(\sin\alpha - f\cos\alpha)$$

Because of vibration, friction may not be very dependable and it is conservative to assume that the axial force to hold the parts together is the larger value of F: $F = F_n \sin\alpha$.

AXIAL FORCE REQUIRED TO DISENGAGE CUP AND CONE. Ordinarily, with the cone angles commonly used, no force is necessary to disengage the parts, although it is possible that if $f\cos\alpha > \sin\alpha$, an axial force F_d may be necessary to disengage the parts:

$$F_d = F_n(f\cos\alpha - \sin\alpha)$$

HORSEPOWER CAPACITY FOR A CONE CLUTCH is as follows, depending upon whether uniform wear or uniform pressure is assumed:

<u>Uniform Wear:</u> $$\text{Hp} = \frac{TN}{63{,}000} = \frac{(F_n f D_m/2)N}{63{,}000} = \frac{FfR_m}{\sin\alpha}\left(\frac{N}{63{,}000}\right)$$

<u>Uniform Pressure:</u> $$\text{Hp} = \frac{TN}{63{,}000} = F_n f\left(\frac{2}{3}\right)\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)\frac{N}{63{,}000} = \frac{Ff}{\sin\alpha}\left(\frac{2}{3}\right)\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)\frac{N}{63{,}000}$$

where F = axial force, lb
f = coefficient of friction
R_o = outside radius, in.
R_i = inside radius, in.
N = speed, rpm
α = cone pitch angle
$F_n = p(2\pi R_m)(b)$, where p is the average pressure, R_m is the mean cone radius, and b is the face width.

SOLVED PROBLEMS

1. Derive the torque capacity for *one pair* of surfaces pressed together with an axial force F. Assume uniform pressure. Refer to Fig. 14-3 below.

Solution:

Consider a differential area $dA = 2\pi r\,dr$. The differential normal force $= dN = p\,dA = p(2\pi r\,dr)$. The differential frictional force $= dQ = f\,dN = f(p 2\pi r\,dr)$. The differential frictional torque $= dT = r\,dQ = r(fp2\pi r\,dr)$. Integrating with p and f as constants, the total torque is

$$T \;=\; 2\pi fp\int_{R_i}^{R_o} r^2\,dr \;=\; 2\pi fp\,\left(\frac{R_o^3 - R_i^3}{3}\right)$$

The axial force $F = p(\pi)(R_o^2 - R_i^2)$, from which the average pressure $p = \dfrac{F}{\pi(R_o^2 - R_i^2)}$.

Substituting this value of p into $T = 2\pi fp\left(\dfrac{R_o^3 - R_i^3}{3}\right)$, we obtain $T = Ff\left[\dfrac{2}{3}\left(\dfrac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right)\right] = FfR_f$

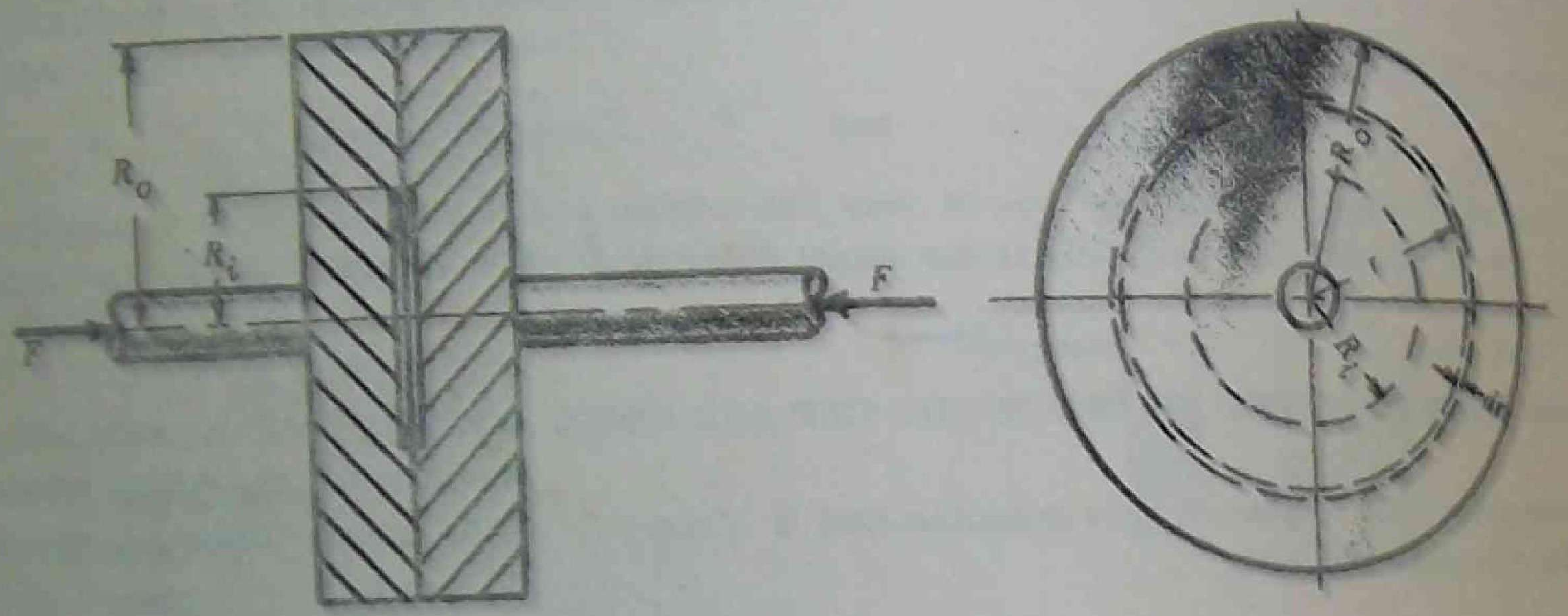


Fig. 14-3

2. Derive the torque capacity for *one pair* of surfaces pressed together with an axial force F. Assume uniform wear. Refer to Fig. 14-3 above.

Solution:

When a clutch is new, it is perhaps true that the pressure may be rather uniform. If the surfaces are relatively rigid, the outer portion, where the velocity is high, will wear more than the inner portion. After initial wearing-in, it is reasonable to assume that the curve of the profile will maintain its shape; or, the wear thereafter may be considered to be uniform.

Uniform wear can be expressed in a different way by saying that in any time interval the work done per unit area is constant:

$$\frac{\text{(frictional force)(velocity)}}{\text{area}} \;=\; \frac{(fp2\pi r\,dr)(r\omega)}{2\pi r\,dr} \;=\; \text{constant } C'$$

or $p = C'/fr\omega$. Then since f and ω are constants, $p = C/r$, where C is a constant.

An alternate method of showing that pressure varies inversely as the radius is to consider that wear δ is proportional to pressure p and velocity V. Thus $\delta = KpV = Kp(r\omega)$, or $p = C/r$ since δ and K are constants and ω is fixed for a given clutch.

As in Problem 1, the differential frictional torque $dT = r(fp2\pi r\,dr)$; the total torque is

$$T \;=\; \int_{R_i}^{R_o} rf\left(\frac{C}{r}\right)2\pi r\,dr \;=\; 2\pi fC\left(\frac{R_o^2 - R_i^2}{2}\right)$$

To find C: $F = \displaystyle\int_{R_i}^{R_o} p(2\pi r\,dr) = \int_{R_i}^{R_o} (C/r)(2\pi r\,dr) = 2\pi C(R_o - R_i)$, or $C = \dfrac{F}{2\pi(R_o - R_i)}$

Substituting this value of C in the equation for T, $T = Ff\left(\dfrac{R_o + R_i}{2}\right) = FfR_f$.

3. Compare the friction radius based upon uniform pressure and uniform wear for two cases: (*1*) $R_o = 4$ in., $R_i = 3.5$ in.; (2) $R_o = 4$ in., $R_i = 1$ in.

Solution:

(*1*) Uniform pressure, $R_f = \frac{2}{3}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right) = \frac{2}{3}\left(\frac{4^3 - 3.5^3}{4^2 - 3.5^2}\right) = 3.76$ in. Uniform wear, $R_f = \frac{R_o + R_i}{2} = \frac{4 + 3.5}{2} = 3.75$ in.

(2) Uniform pressure, $R_f = \frac{2}{3}\left(\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}\right) = \frac{2}{3}\left(\frac{4^3 - 1^3}{4^2 - 1^2}\right) = 2.8$ in. Uniform wear, $R_f = \frac{R_o + R_i}{2} = \frac{4+1}{2} = 2.5$ in.

Thus for low values of R_o/R_i the difference between uniform wear and uniform pressure is very small. As R_o/R_i increases, the difference becomes larger.

4. Plot the ratio of friction radius to outside radius (R_f/R_o) versus the ratio of inside radius to outside radius (R_i/R_o) for uniform pressure and uniform wear assumptions. Refer to Fig. 14-4 for solution.

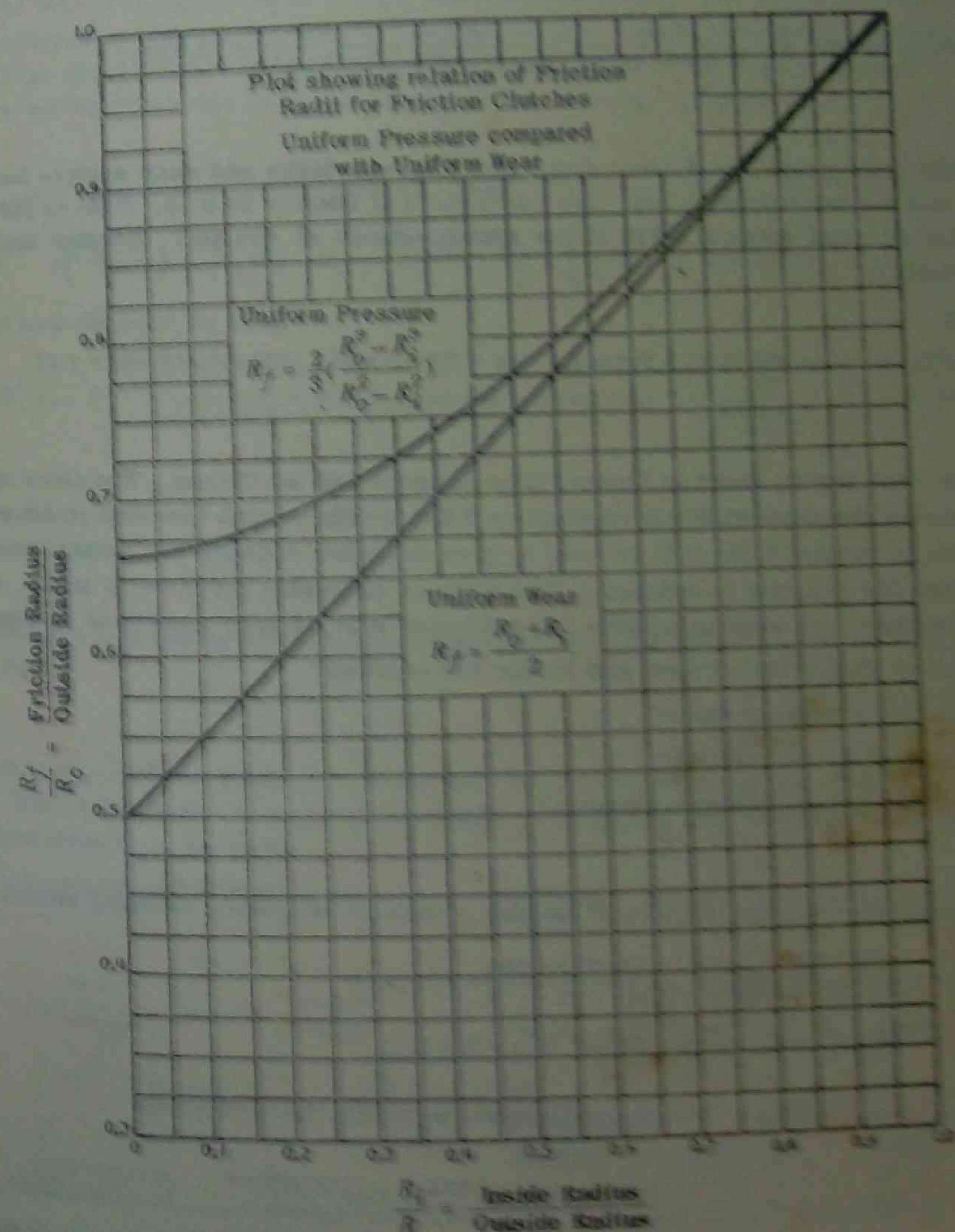


Fig. 14-4

15

Brake Design

BRAKES are machine elements that absorb either kinetic or potential energy in the process of slowing down or stopping a moving part. The absorbed energy is dissipated as heat. Brake capacity depends upon the unit pressure between the braking surfaces, the coefficient of friction, and the ability of the brake to dissipate heat equivalent to the energy being absorbed. The performance of brakes is similar to that of clutches except that clutches connect one moving part to another moving part, whereas brakes connect a moving part to a frame.

EXTERNAL SHOE OR BLOCK BRAKES consist of shoes or blocks pressed against the surface of a rotating cylinder called the brake drum. The shoe may be rigidly mounted to a pivoted lever as shown in Fig. 15-1, or the shoe may be pivoted to the lever as shown in Fig. 15-2.

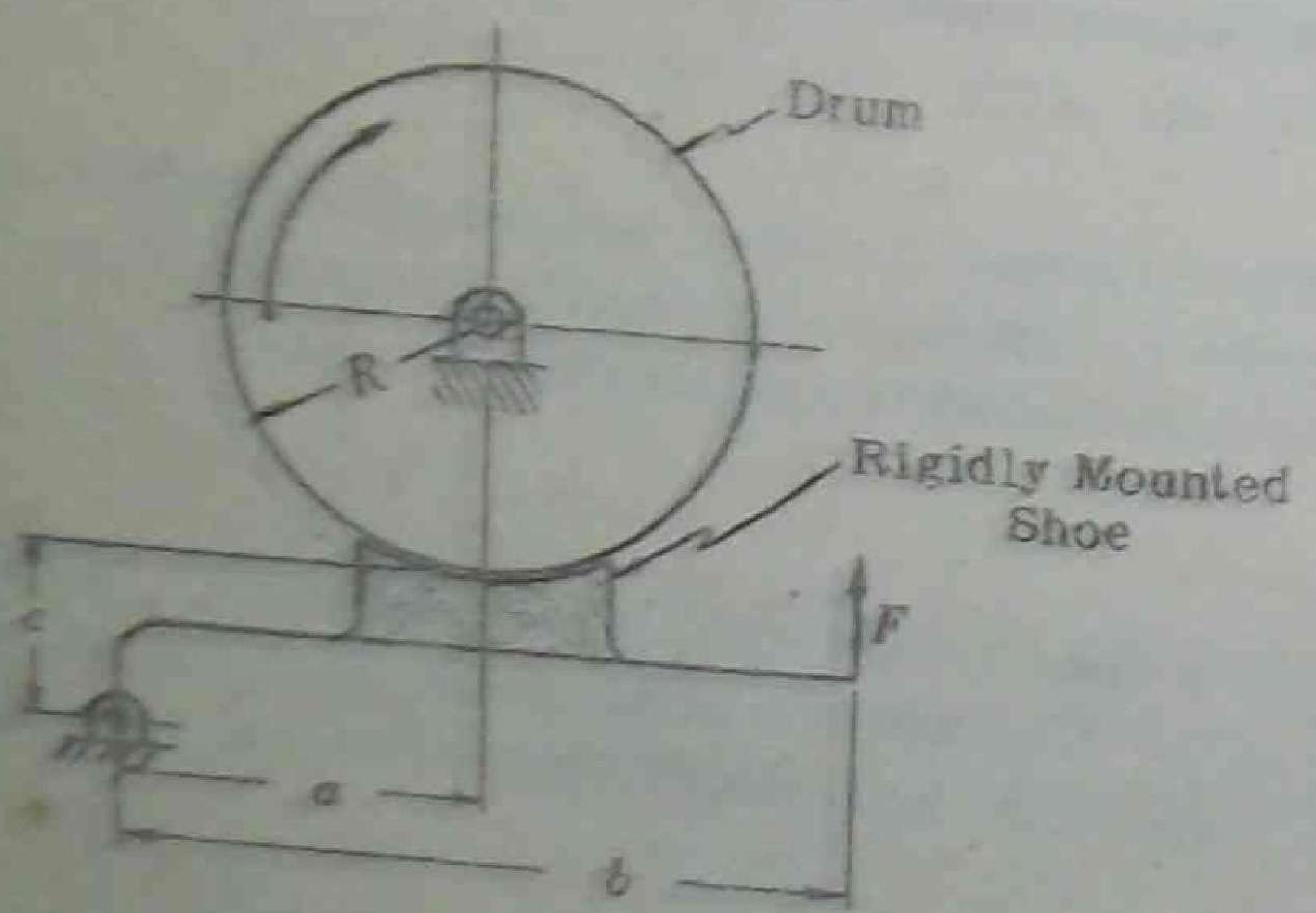


Fig. 15-1

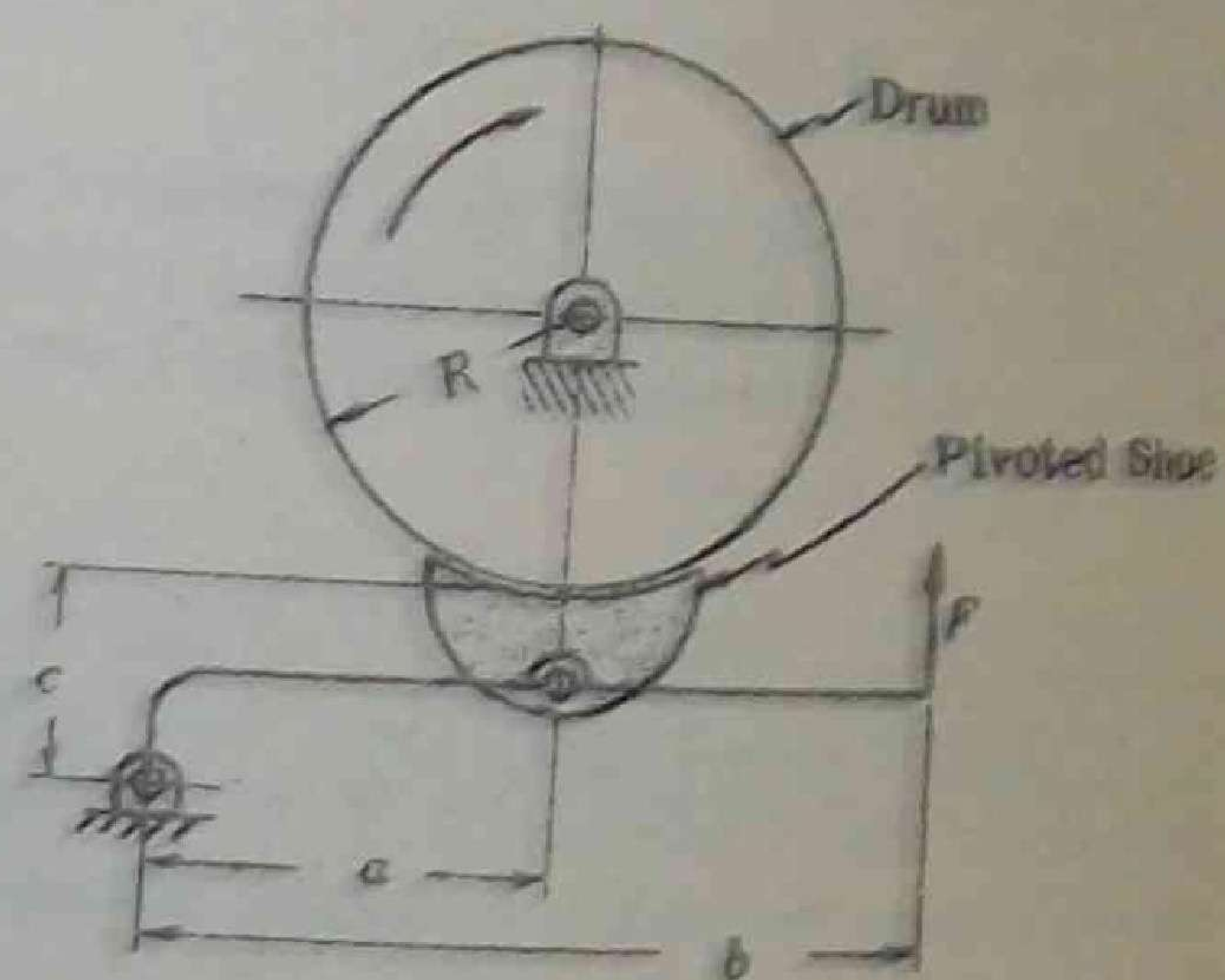


Fig. 15-2

SINGLE BLOCK BRAKE design may be based on the force and torque analysis of the lever and shoe as a free body, as shown in Fig. 15-3. The normal force N and the frictional force fN may be shown acting at the midpoint of contact of the shoe without appreciable error for angles of θ not greater than 60°. Summing moments about the fixed pivot O,

$$(N+W)a - fNc - Fb = 0 \quad \text{or} \quad F = \frac{(N+W)a - fNc}{b}$$

Note that for a clockwise rotation of the drum, the friction force fN aids the force F in applying the brake and the brake is partially self-actuating. For a given coefficient of friction the brake may be designed to be wholly self-actuating (or self-locking). For this condition to exist, $F = 0$ or negative in the above equa-

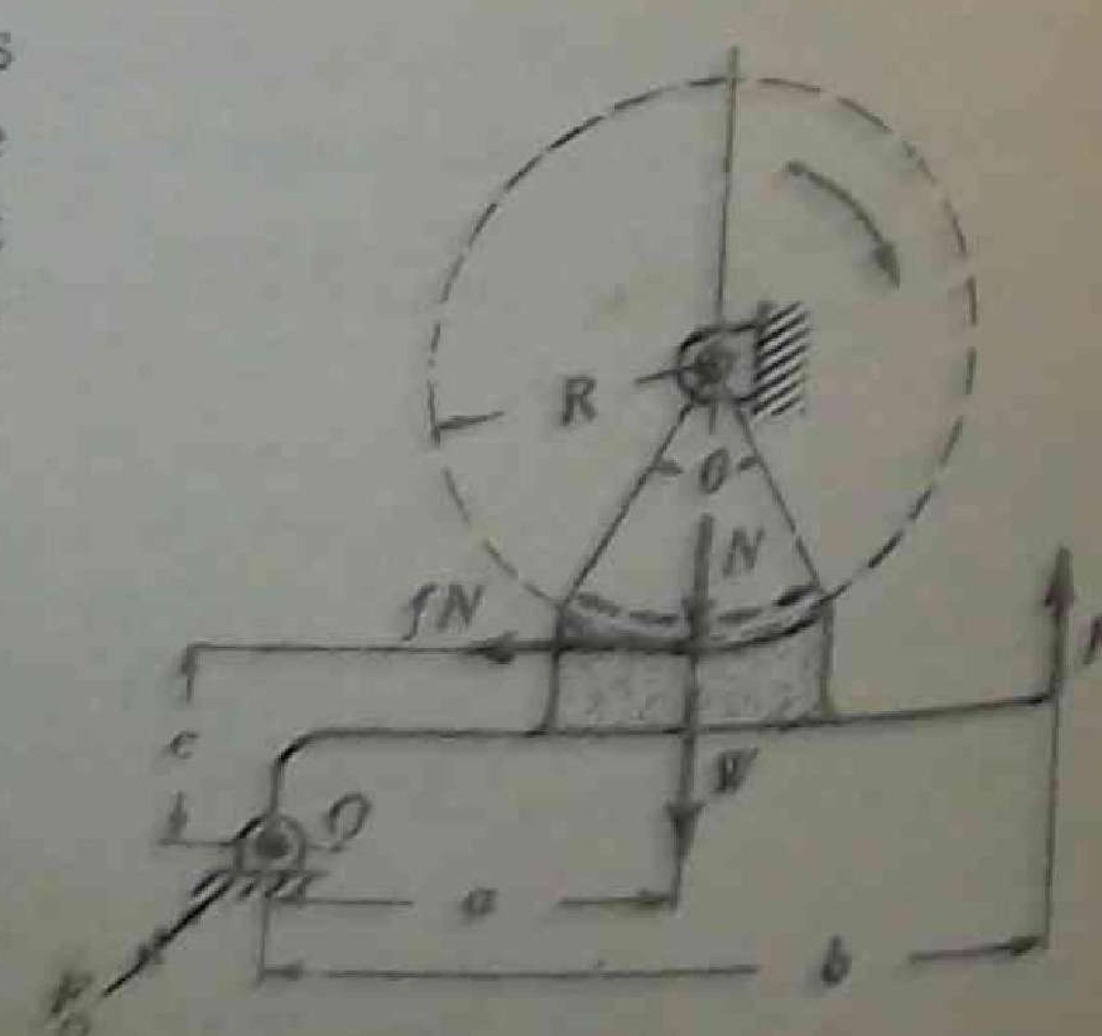


Fig. 15-3

tion. We may also assume that the weight F is negligible; then

$$F = \frac{Na - fNc}{b} \leq 0$$

i.e. when $\frac{a}{c} \leq f$ the brake is self-locking.

The braking torque T for a non-self-locking situation is

$$T = fNR \text{ in-lb}$$

where f = coefficient of friction; N = total normal force, lb; R = radius of brake drum, in.

When the angle of contact is large, say 60° or more, an appreciable error might result from assuming that the frictional and normal forces act at the midpoint of contact of the shoe. A more precise analysis shows that the frictional force fN is shifted away from the surface of the drum to a point D at a distance h from the center of the drum as shown in Fig. 15-4 for a *pivoted* shoe. The pivoted shoe is the more usual construction when using long external shoes. The braking torque T is now

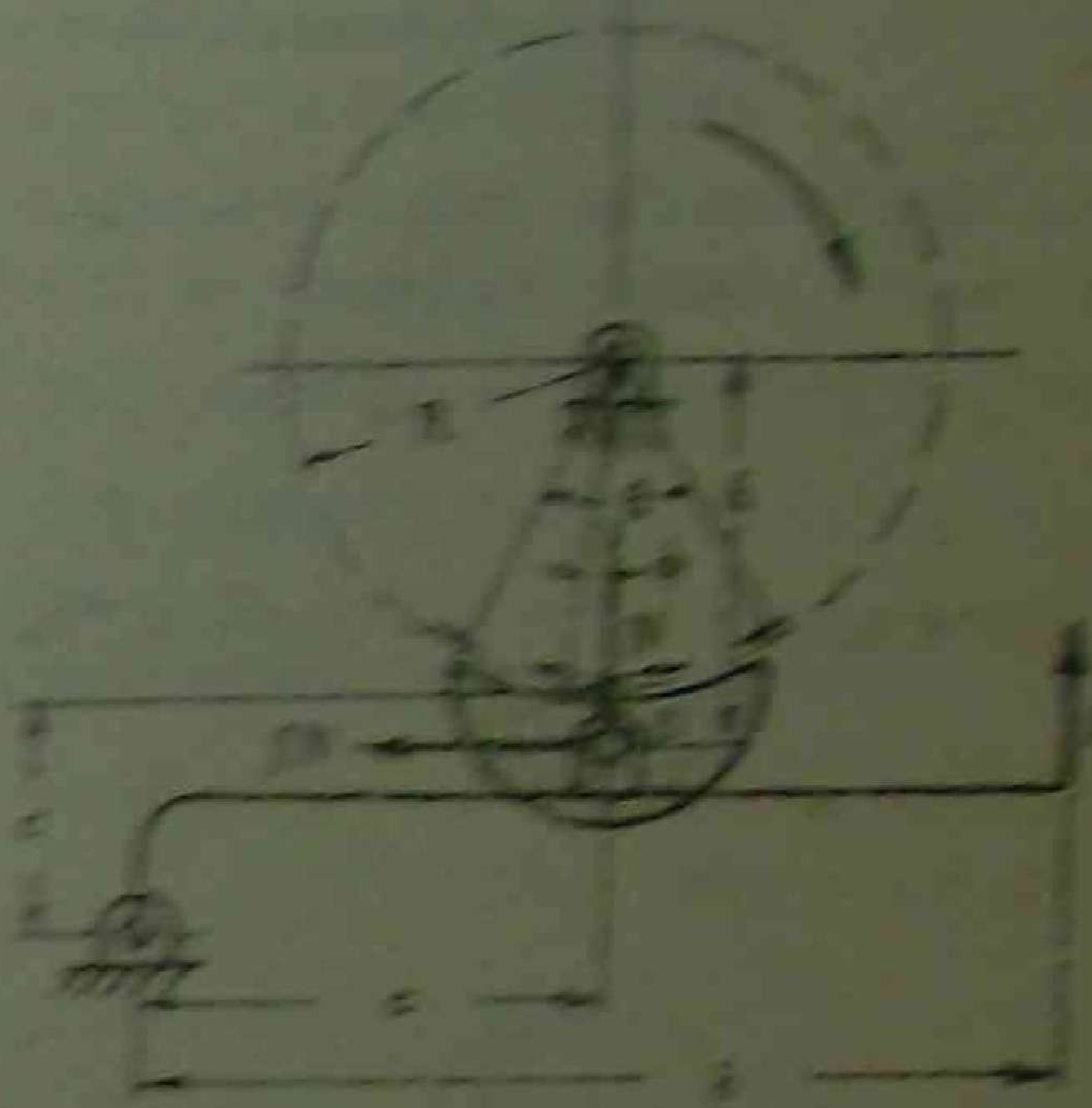


Fig. 15-4

$$T = fNh = fN\left(\frac{4R\sin\frac{1}{2}\theta}{\theta + \sin\theta}\right) \quad \text{(See Problem 5.)}$$

where $h = \dfrac{4R\sin\frac{1}{2}\theta}{\theta + \sin\theta}$. This is based on assuming that wear in the direction of the resultant normal force N is uniform, which means that the normal pressure p_n varies as the cosine of the angle ϕ, or

$$p_n = C\cos\phi$$

where C is a constant $= \dfrac{2N}{wR(\theta + \sin\theta)}$, w being the width of the brake shoe in inches.

The magnitude of h then determines the location of the pivot for the pivoted shoe. Two conditions have been satisfied: (1) the shoe is a two force member, and the resultant of the normal force and frictional force must pass through the pivot; (2) the pressure distribution is as assumed. If the pivot of the shoe is located at a distance other than h, as calculated, the moment of the resultant normal force and frictional force would still be zero about the pivot, but now the assumed pressure distribution cannot exist. Consequently the pressure must change and greater wear will occur at either the trailing or leading edge. However, if the pivot is located at a small distance from the theoretical value of h, based upon the pressure distribution $p_n = C\cos\phi$, the above equations may still be used without appreciable error. Also the pivot may be located anywhere along the resultant of the frictional and normal forces without affecting the assumed pressure distribution.

The average pressure, $p_{av} = \dfrac{2C\sin\frac{1}{2}\theta}{\theta}$

DOUBLE SHOE BRAKES are commonly used in order to reduce shaft and bearing loads, to obtain greater capacity, and to reduce the amount of heat generated per sq in. See Fig. 15-5. The normal force N_L on the left shoe is not necessarily equal to the normal force N_R on the right shoe. For double block brakes whose shoes have small contact angles, of say less than 60°, the braking torque may be approximated by

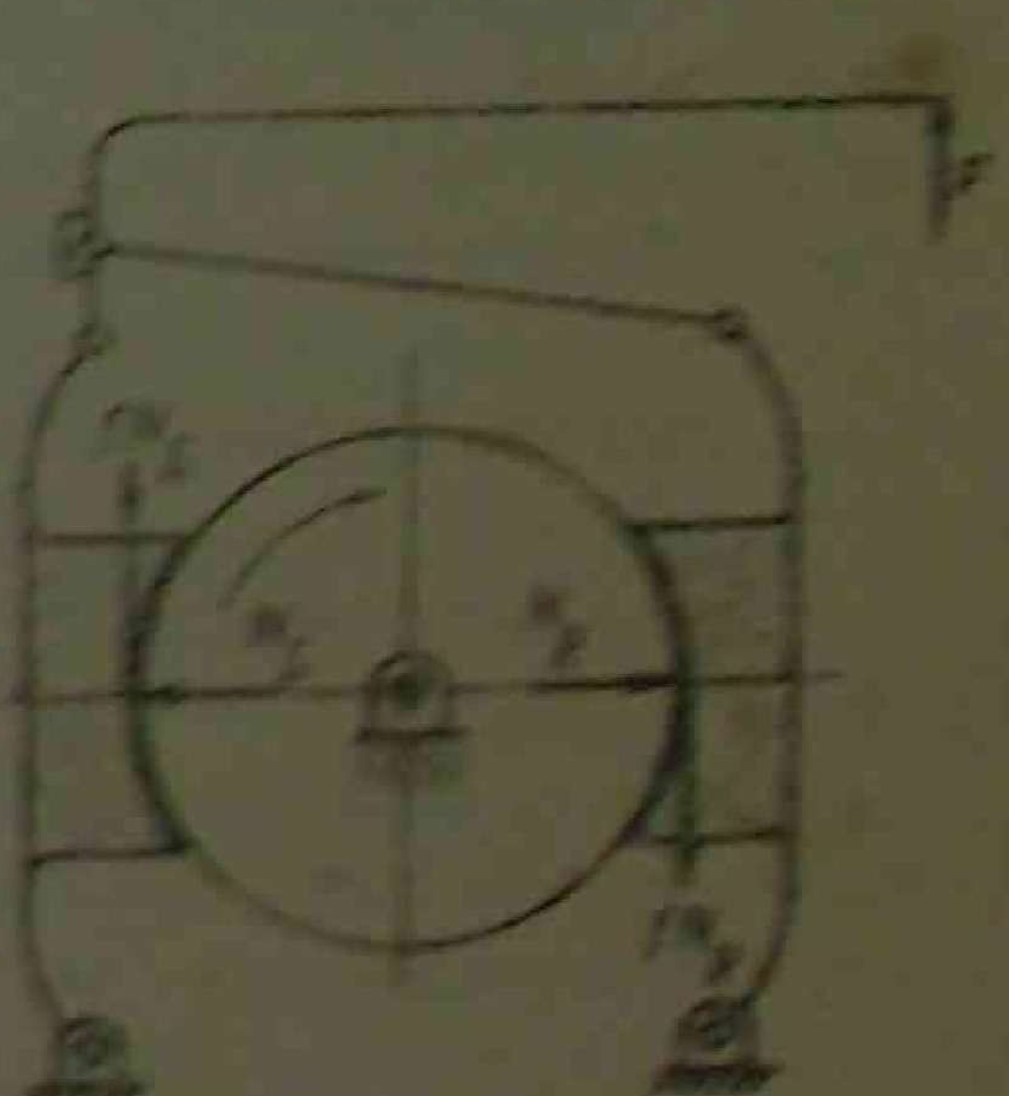


Fig. 15-5

$$T = f(N_L + N_R)R$$

If the shoe contact angle is greater than about 60°, then a more precise evaluation of the braking torque for pivoted shoes is

$$T = f(N_L + N_R)\left(\frac{4R\sin\frac{1}{2}\theta}{\theta + \sin\theta}\right)$$

If long rigid shoes are to be used, an analysis similar to that for internal shoes as presented on the following page would apply.

INTERNAL SHOE BRAKE design of the symmetrical type shown in Fig. 15-6 may be approximated by the equations given below.

The braking torque T may be determined by

$$T = fwr^2\left(\frac{\cos\theta_1 - \cos\theta_2}{\sin\theta_m}\right)(p_m + p'_m)$$

where f = coefficient of friction

w = face width of shoe, in.

r = internal radius of drum, in.

θ_1 = center angle from shoe pivot to heel of lining, degrees

θ_2 = center angle from shoe pivot to toe of lining, degrees

p_m = maximum pressure, psi (right shoe)

p'_m = maximum pressure, psi (left shoe) $= \dfrac{cFp_m}{M_n + M_f}$

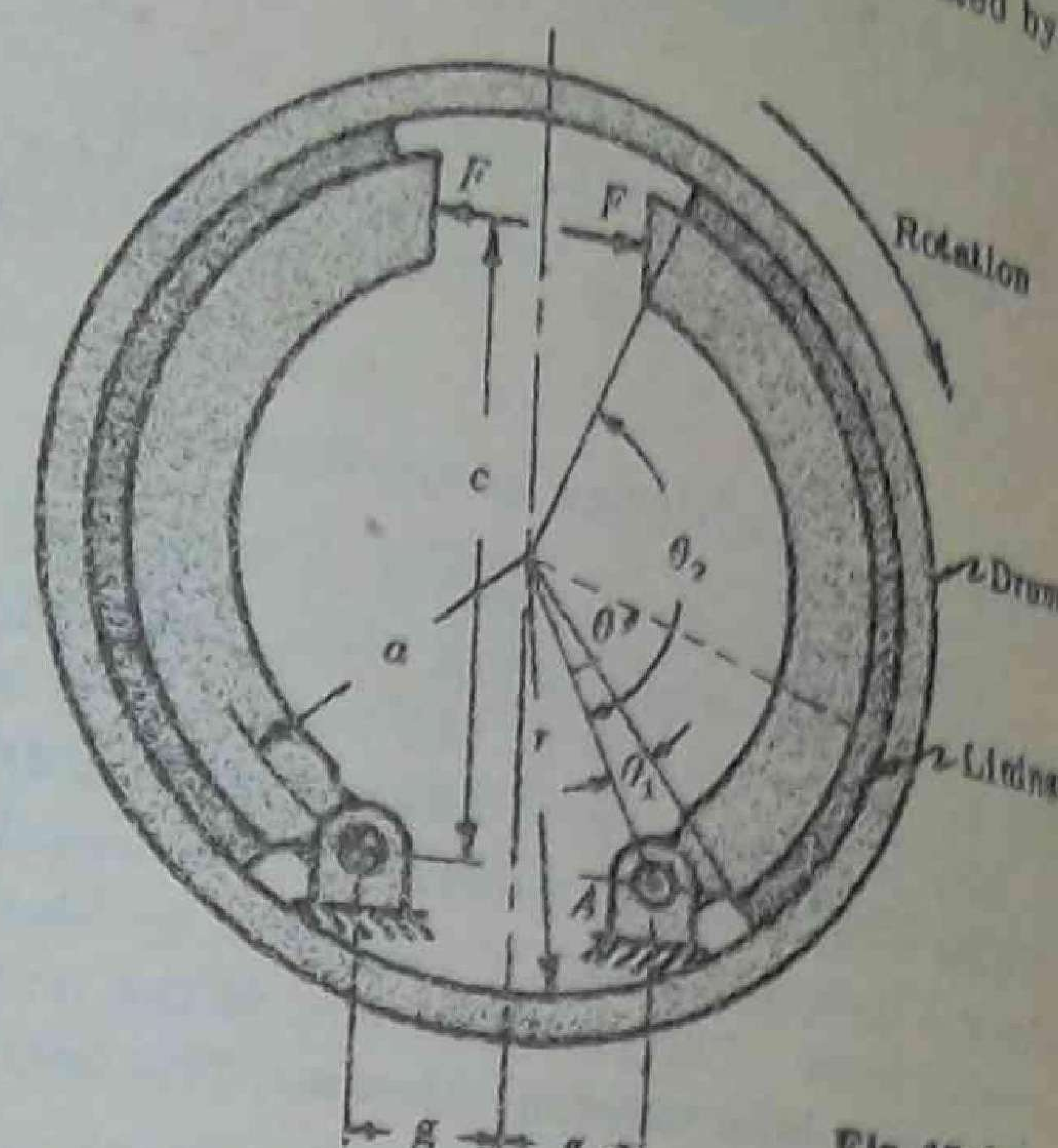


Fig. 15-6

The above is based on the assumed pressure distribution

$$p = p_m \frac{\sin\theta}{\sin\theta_m}$$

where θ_m = center angle from shoe pivot to point of maximum pressure, psi

$\theta_m = 90°$ if $\theta_2 > 90°$, $\theta_m = \theta_2$ if $\theta_2 < 90°$.

The moment M_f of the frictional forces with respect to the shoe pivot may be determined by

$$M_f = \frac{fp_m wr}{\sin\theta_m}\int_{\theta_1}^{\theta_2} \sin\theta(r - a\cos\theta)\,d\theta$$

where a = distance from the center of the drum to the shoe pivot, in.

The moment M_n of the normal forces with respect to the shoe pivot may be determined by

$$M_n = \frac{p_m wra}{\sin\theta_m}\int_{\theta_1}^{\theta_2} \sin^2\theta\,d\theta$$

The actuating force F may be determined by setting the summation of moments about the pin joint equal to zero. For clockwise rotation of the drum, the right shoe has self actuating properties and $F = \dfrac{M_n - M_f}{c}$; for the left shoe, $F = \dfrac{M'_n + M'_f}{c}$

where c = the moment arm, in inches, of the actuating force F, $M'_n = \dfrac{M_n p'_m}{p_m}$, $M'_f = \dfrac{M_f p'_m}{p_m}$.

The above equations are based on the following assumptions:

1) The normal pressure at any contact point on the shoe is proportional to its vertical distance from the pivot point.
2) The shoe is rigid.
3) The coefficient of friction does not vary with pressure and velocity.

BAND BRAKES consist of a flexible band wrapped partly around the drum. They are actuated by pulling the band tightly against the drum. The brake [illegible] wrap, the coefficient of friction, and the band tension [illegible] below. For this type of brake the [illegible] the frame is the tight side [illegible]

As for belts at zero velocity, the relationship between the tight and loose sides of the band is

$$F_1/F_2 = e^{f\alpha}$$

where F_1 = tension in tight side of band, lb
F_2 = tension in loose side of band, lb
e = natural logarithm base
f = coefficient of friction
α = angle of wrap, radians.

The torque braking capacity T is

$$T = (F_1 - F_2)r \text{ in-lb}$$

where r = radius of brake drum, in. This type of band brake does not have self-actuating properties.

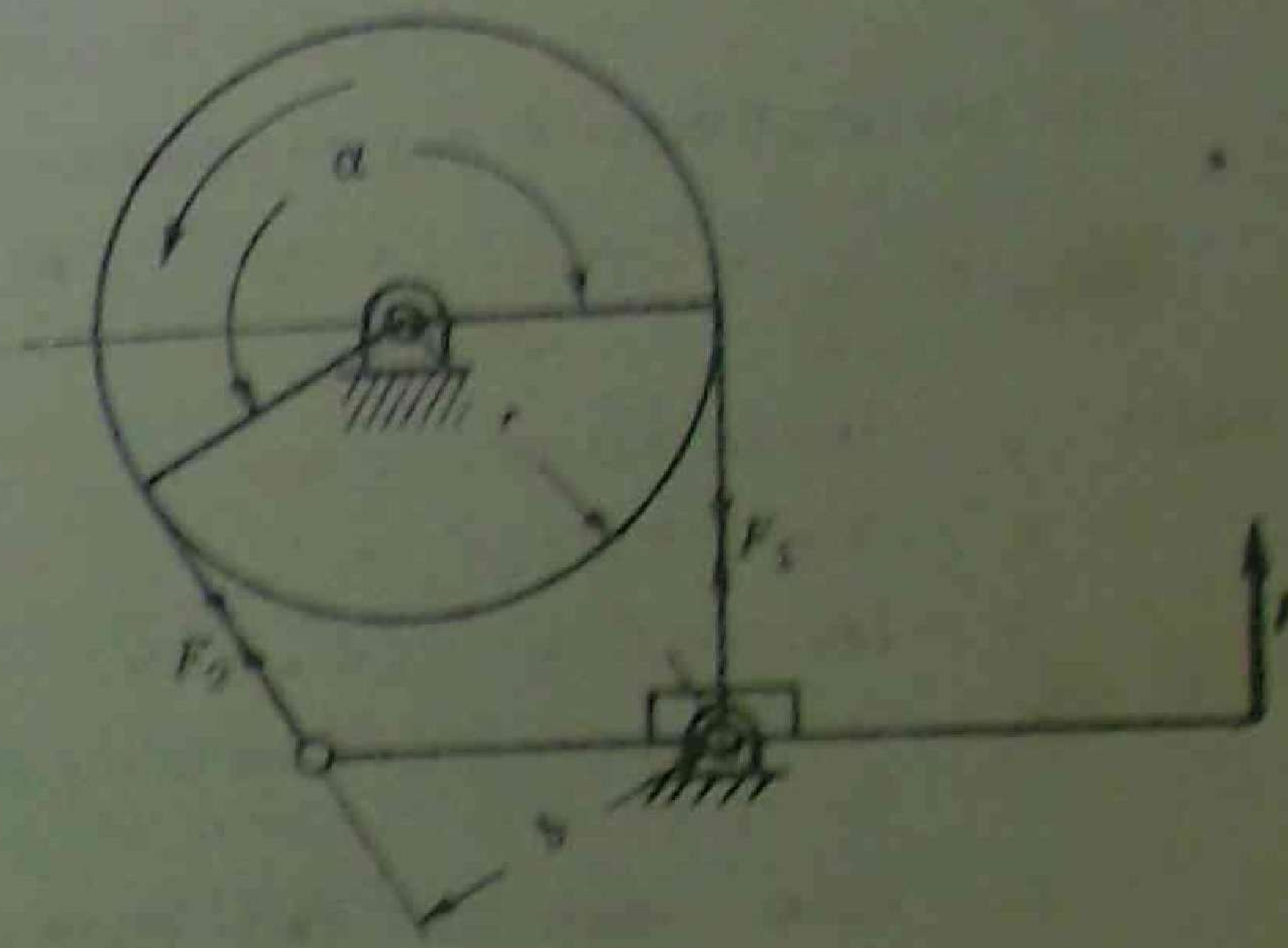


Fig. 15-7

A simple two-way band brake is shown in Fig. 15-8 below. This type of design functions equally well for either direction of rotation since the moment arms of the tight and loose tensions are equal.

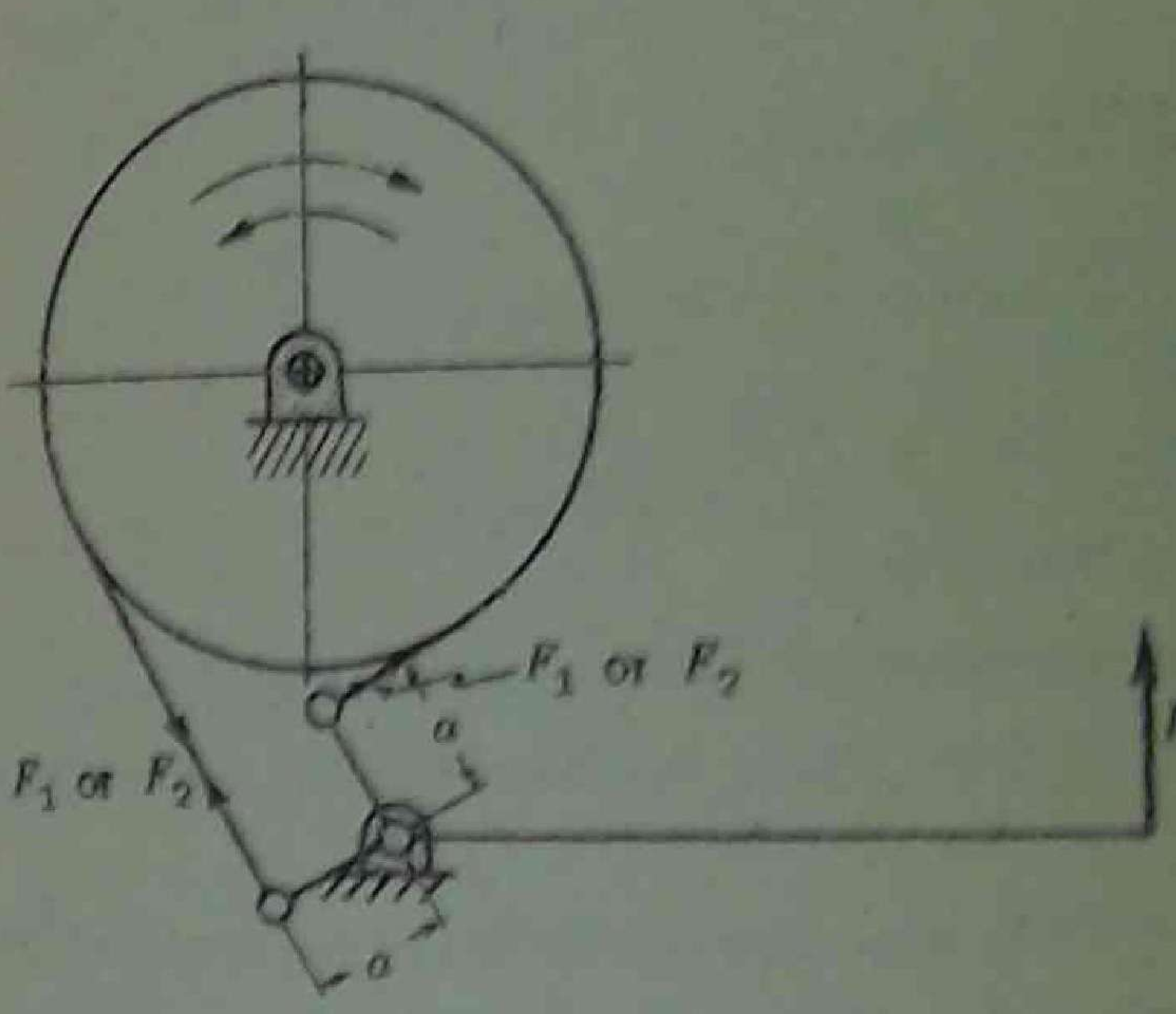


Fig. 15-8

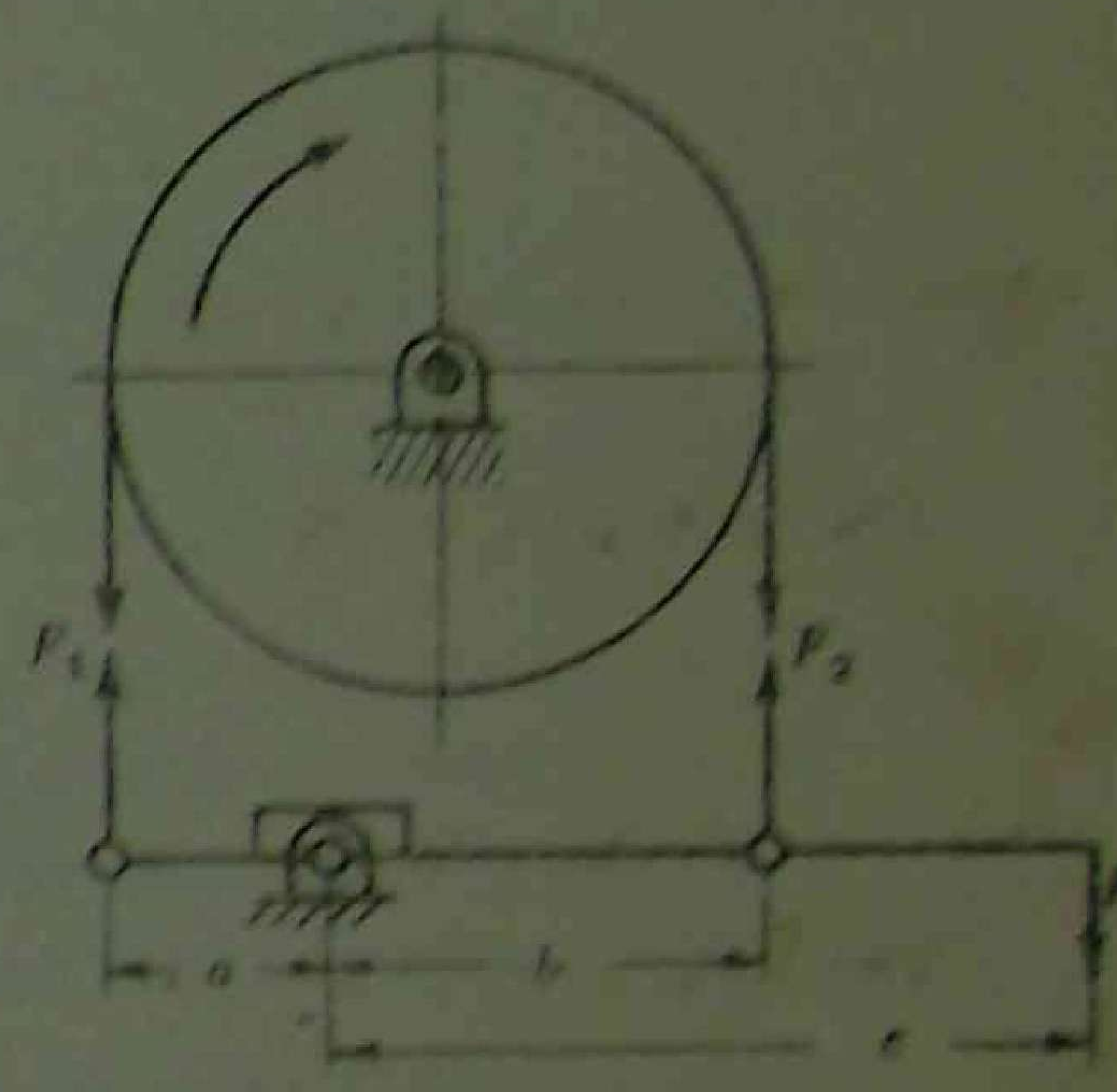


Fig. 15-9

The differential band brake as shown in Fig. 15-9 above is one that has self-actuating properties and may be designed to be self-locking. The differential band brake is usually designed so that the direction of drum rotation permits the tight side of the band to aid in applying the brake. Referring to Fig. 15-9, if we take the summation of moments with respect to the pivot, we have

$$Fc + F_1 a - F_2 b = 0 \quad \text{or} \quad F = \frac{F_2 b - F_1 a}{c}$$

Substituting $F_1 = F_2 e^{f\alpha}$

$$F = \frac{F_2(b - e^{f\alpha}a)}{c}$$

from which it can be seen that for a self-locking brake, i.e. when $F = 0$ or negative,

$$b \le ae^{f\alpha} \quad \text{or} \quad b/a \le e^{f\alpha}$$

It should be noted that the differential band brake may be made self-locking for one direction of rotation only. A self-locking brake of this type is used to allow motion in one direction only and to prevent a reversed motion, as might occur when a conveyor or hoist is acted upon by gravity. If a brake is self-locking, it requires a force in the opposite direction of applying the brake in order to have it released. Also, after the brake has locked and additional torque is applied, the band tensions F_1 and F_2 will increase, but the ratio of F_1 to F_2 will no longer equal $e^{f\alpha}$ since this relationship prevails only when the brake is slipping or when slip is pending.

The maximum unit pressure p_m occurs at the tight end of the band and is determined by

$$p_m = \frac{F_1}{w\,r}$$

The average normal pressure between the band and the drum (which is used in heat generated calculations) is

$$p_{av} = \frac{F_1}{w\,r\,f\alpha}\left(\frac{e^{f\alpha}-1}{e^{f\alpha}}\right)$$

HEAT GENERATED during the application of a brake must be dissipated by heat transfer or the brake will overheat and perhaps burn out the lining. The rate of heat generated, H_g, is equal to the rate of frictional work:

$$H_g = p_{av} A_c f V/778 \text{ Btu/min}$$

where p_{av} = average contact pressure, psi
A_c = contact area, in^2
f = coefficient of friction
V = peripheral velocity of drum, ft/min.

The heat generated may also be determined by considering the amount of kinetic or potential energy that is being absorbed:

$$H_g = (E_p + E_k)/778 \text{ Btu/min}$$

where E_p = total potential energy absorbed, ft-lb/min
E_k = total kinetic energy absorbed, ft-lb/min.

The heat dissipated, H_d, may be estimated by

$$H_d = C \Delta t A_r \text{ Btu/min}$$

where C = coefficient of heat transfer, Btu per in^2 per min per °F temperature difference
Δt = temperature difference between the exposed radiating surface and the surrounding air
A_r = area of radiating surface, in^2.

C may be of the order of 0.0006 for a Δt of 100°F and increase up to 0.0009 for a Δt of 400°F.

The expressions for heat dissipated are quite approximate and should serve only as an indication of the capacity of the brake to dissipate heat. The exact performance of the brake should be determined by test. Another convenient indicator of brake capacity is hp/wd which is limited to about 0.3, where w = width of band or shoe and d = diameter of drum in inches.

Experience has also shown that the product of the average pressure p_{av} (psi of projected area) and the rubbing velocity V (ft/min) should be limited as follows: $p_{av}V \leq 28{,}000$ for continuous application of load, as in lowering operations, and poor dissipation of heat; $p_{av}V \leq 55{,}000$ for intermittent application of load, with comparatively long periods of rest, and poor dissipation of heat; $p_{av}V \leq 83{,}000$ for continuous application of load and good dissipation of heat, as in an oil bath.

Some permissible average values for operating temperatures, coefficient of friction, and maximum contact pressure for brake materials are given below.

Material	Max. Temp., °F	f	p_{max}, psi
Metal on metal	600	0.25	200
Wood on metal	150	0.25	70
Leather on metal	150	0.35	25
Asbestos on metal in oil	500	0.40	50
Sintered metal on cast iron in oil	500	0.15	400

SOLVED PROBLEMS

1. A 14 in. radius brake drum contacts a single shoe as shown in Fig. 15-10 below and sustains 2000 in-lb of torque at 500 rpm. For a coefficient of friction of 0.3 determine:

(a) The total normal force N on the shoe.

(b) The required force F to apply the brake for clockwise rotation.

(c) The required force F to apply the brake for counterclockwise rotation.

(d) The dimension c required to make the brake self-locking, assuming the other dimensions remain as shown.

(e) The rate of heat generated, Btu/min.

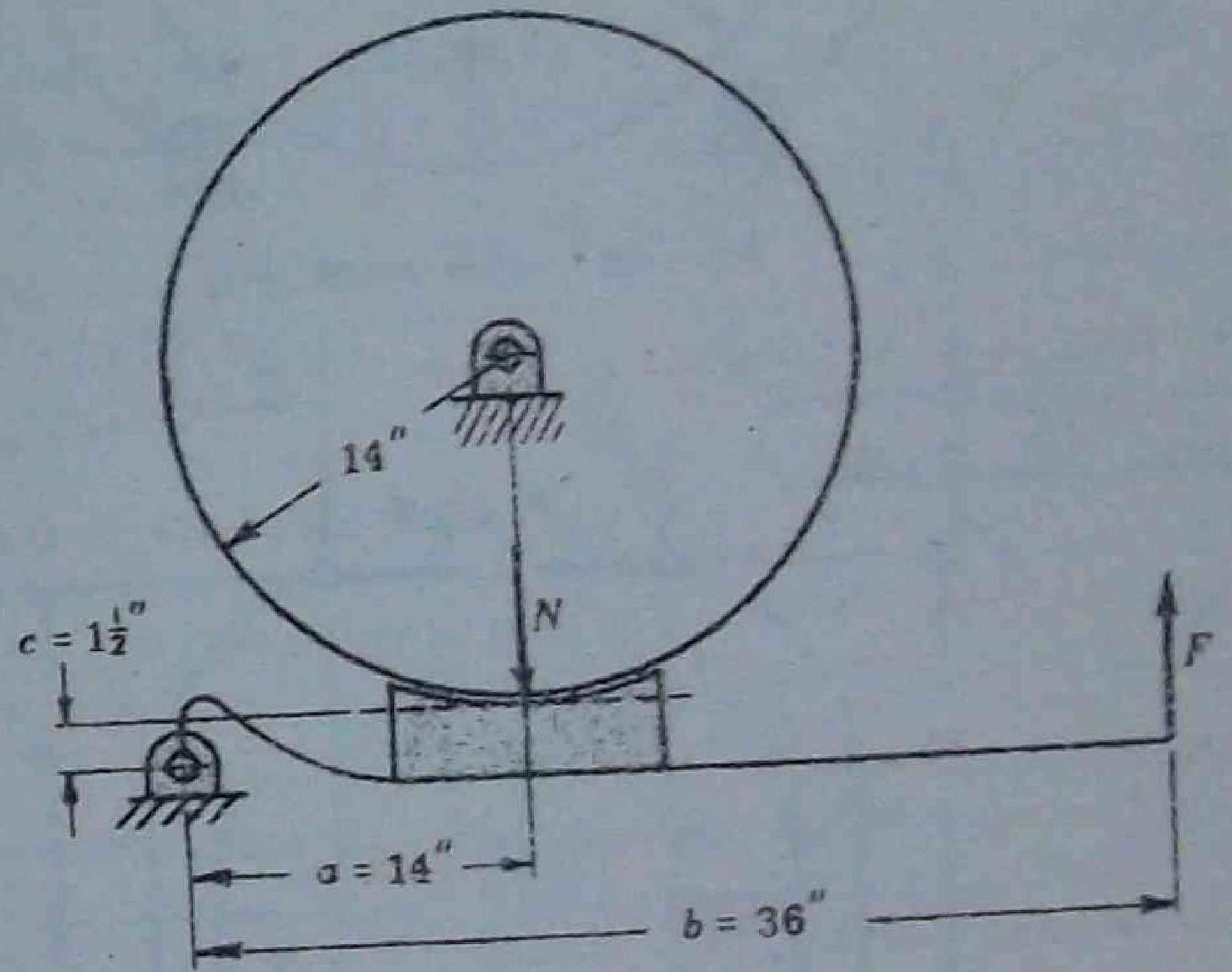


Fig. 15-10

Solution:

(a) Torque $= fNr = 0.3N(14) = 2000, \quad N = 476$ lb

Frictional force $= fN = 2000/14 = 143$ lb

(b) For clockwise rotation, take summation of moments about the pivot equal to zero:

$$(1.5)(143) + 36F - (14)(476) = 0, \quad F = 179 \text{ lb}$$

(c) For counterclockwise rotation, take summation of moments about the pivot equal to zero:

$$(14)(476) + (1.5)(143) - 36F = 0, \quad F = 191 \text{ lb}$$

(d) For self-locking, which can only occur for clockwise rotation of the drum,

$$a \leqq fc \quad \text{or} \quad c \geqq a/f = 14/0.3 = 46.7 \text{ in.}$$

(e) $$H_g = \frac{fNV}{778} = \frac{(0.3)(476)[\pi 28(500)/12]}{778} = 674 \text{ Btu/min}$$

Springs

SPRING DESIGN involves the relationship between force, torque, deflection, and stress. Springs have many uses in connection with machine design, such as to cushion impact and shock loading, to store energy, to maintain contact between machine members, for force measuring devices, to control vibration, and other related functions.

MULTI-LEAF SPRINGS may be of either the simple cantilever type as shown in Fig. 16-1(*a*), or the semi-elliptic leaf spring design as shown in Fig. 16-1(*b*). The design of these springs is usually based upon the force, deflection, and stress relationships that apply to beams of constant strength and uniform thickness. Such beams are of triangular profile.

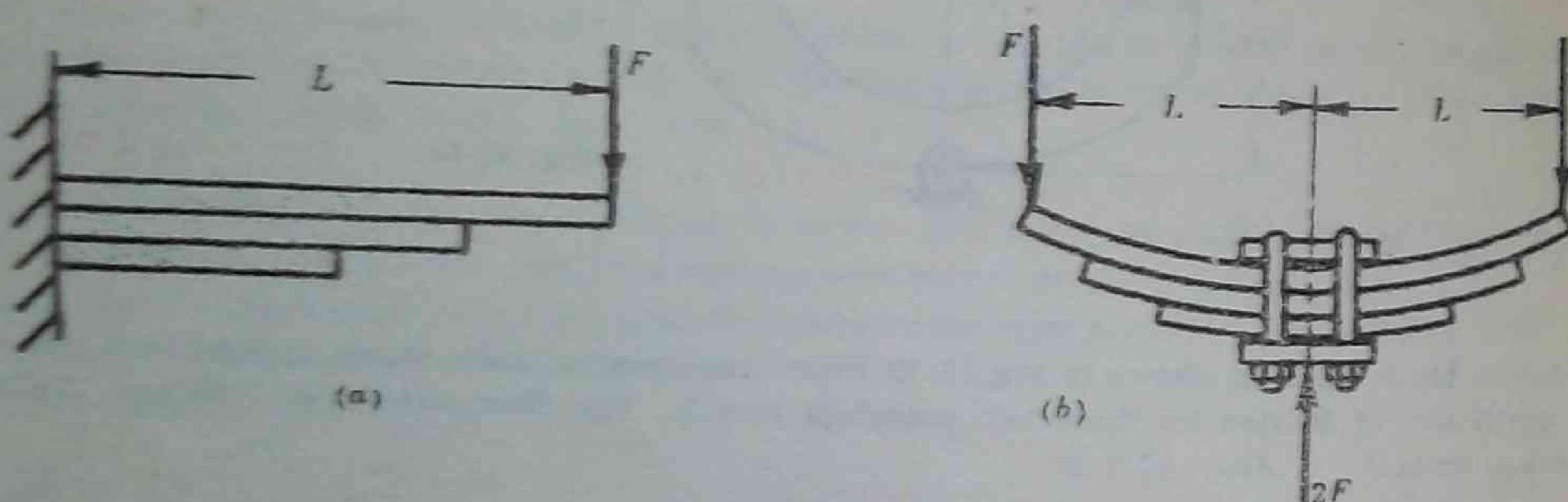


Fig. 16-1

Basically, the multi-leaf spring may be considered as a triangular plate as shown in Fig. 16-2(*a*) cut into *n* strips of width *b*, and stacked in a graduated manner as shown in Fig. 16-2(*b*). A graduated spring made from a triangular beam comes to a point at its end, which is satisfactory from the standpoint of bending stress. However, sufficient metal must be provided to support transverse shear and to provide for load connections which in turn are sometimes called upon to carry end thrust and twisting action. This may be accomplished by adding one or more extra full length leaves, n_e, of uniform width and thickness on top of the graduated stack, as shown in Fig. 16-3 below.

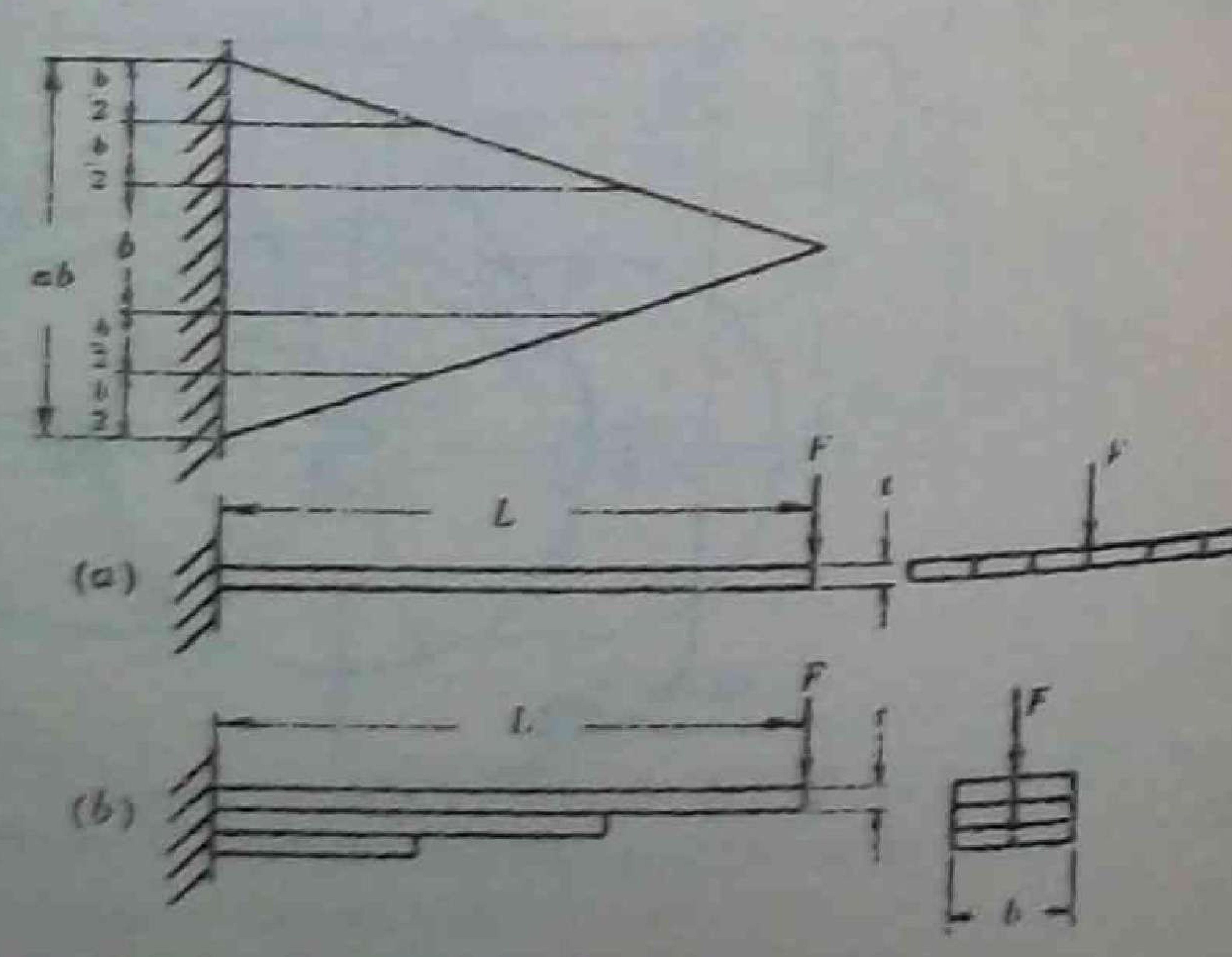


Fig. 16-2

Note that the number of extra full length leaves, n_e, is always one less than the total

Curved Beams

BENDING STRESSES IN CURVED BEAMS do not follow the same linear variation as straight beams, because of the variation in arc length. Even though the same assumptions are used for both types, i.e. that plane sections perpendicular to the axis of the beam remain plane after bending and that stress is proportional to strain, the distribution of stress is quite different. Fig. 4-1 shows the linear stress variation in a straight beam and the hyperbolic stress distribution in a curved beam. Note that the bending stress in the curved beam is zero at a point other than at the center of gravity axis. Also, note that the neutral axis is located between the gravity axis and the center of curvature; this always occurs in curved beams.

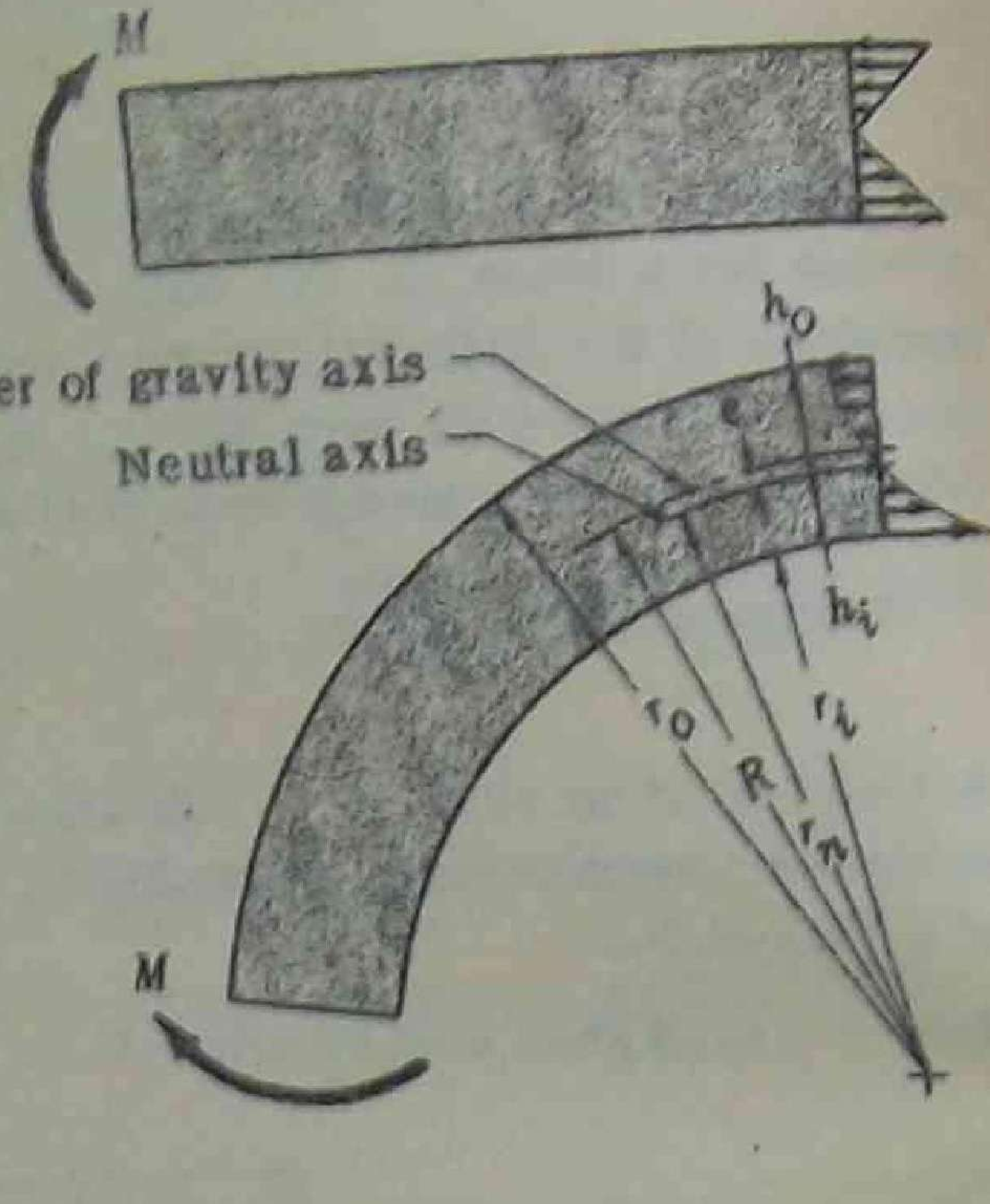


Fig. 4-1

STRESS DISTRIBUTION due to bending is given by $s = \dfrac{My}{Ae(r_n - y)}$

where s is the bending stress, psi

M is the bending moment with respect to the centroidal axis, in-lb

y is the distance from the neutral axis to the point in question, inches (positive for distances toward the center of curvature, negative for distances away from the center of curvature)

A is the area of the section, sq in.

e is the distance from the center of gravity axis to the neutral axis, inches

r_n is the radius of curvature of the neutral axis, inches.

BENDING STRESS AT THE INSIDE FIBER is given by $s = \dfrac{Mh_i}{Aer_i}$

where h_i is the distance from the neutral axis to the inside fiber, inches ($h_i = r_n - r_i$)

r_i is the radius of curvature of the inside fiber, inches.

BENDING STRESS AT THE OUTSIDE FIBER is given by $s = \dfrac{Mh_o}{Aer_i}$

where h_o is the distance from the neutral axis to the outside fiber, inches ($h_o = r_o - r_n$)

r_o is the radius of curvature of the outside fiber, inches.

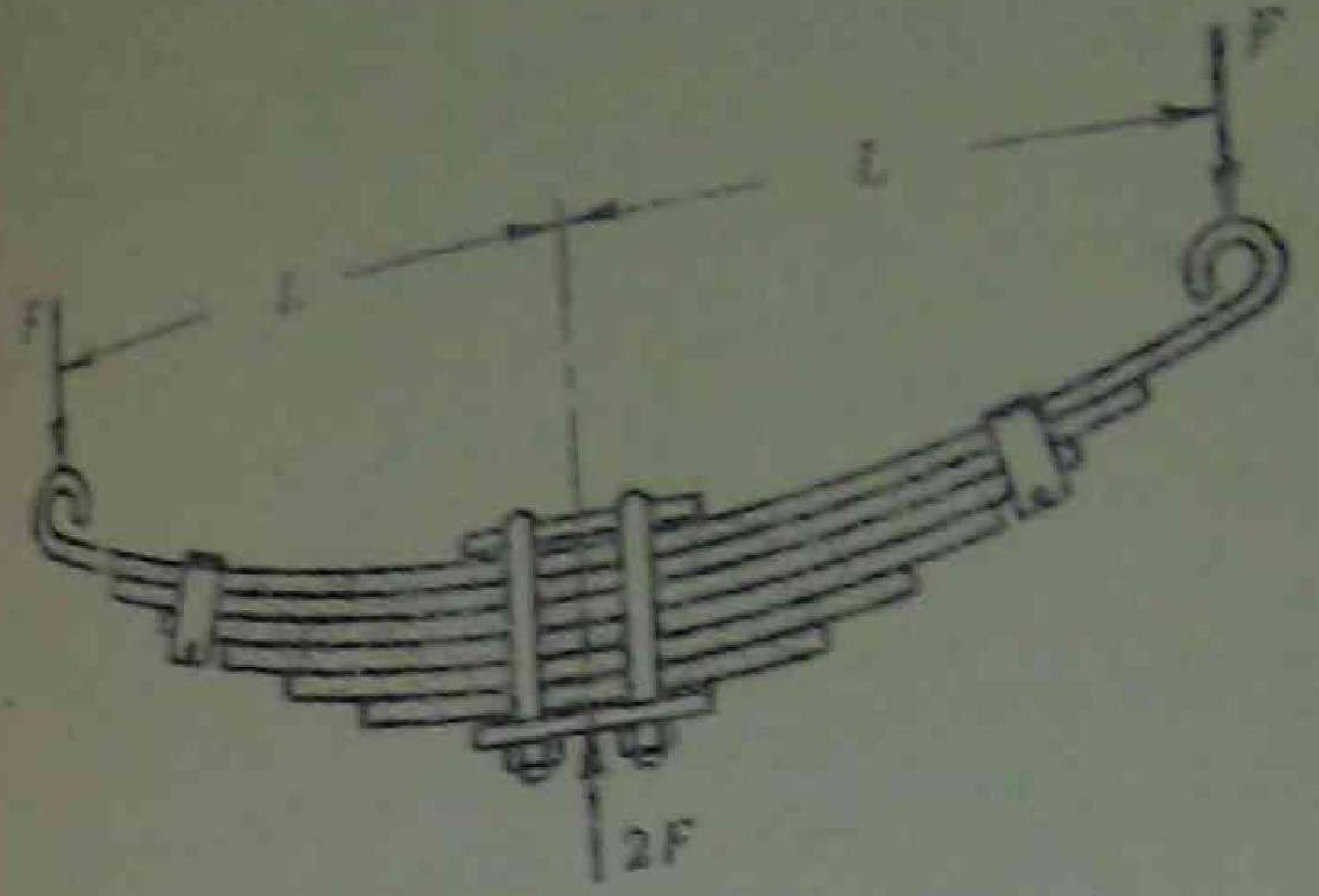


Fig. 16-3

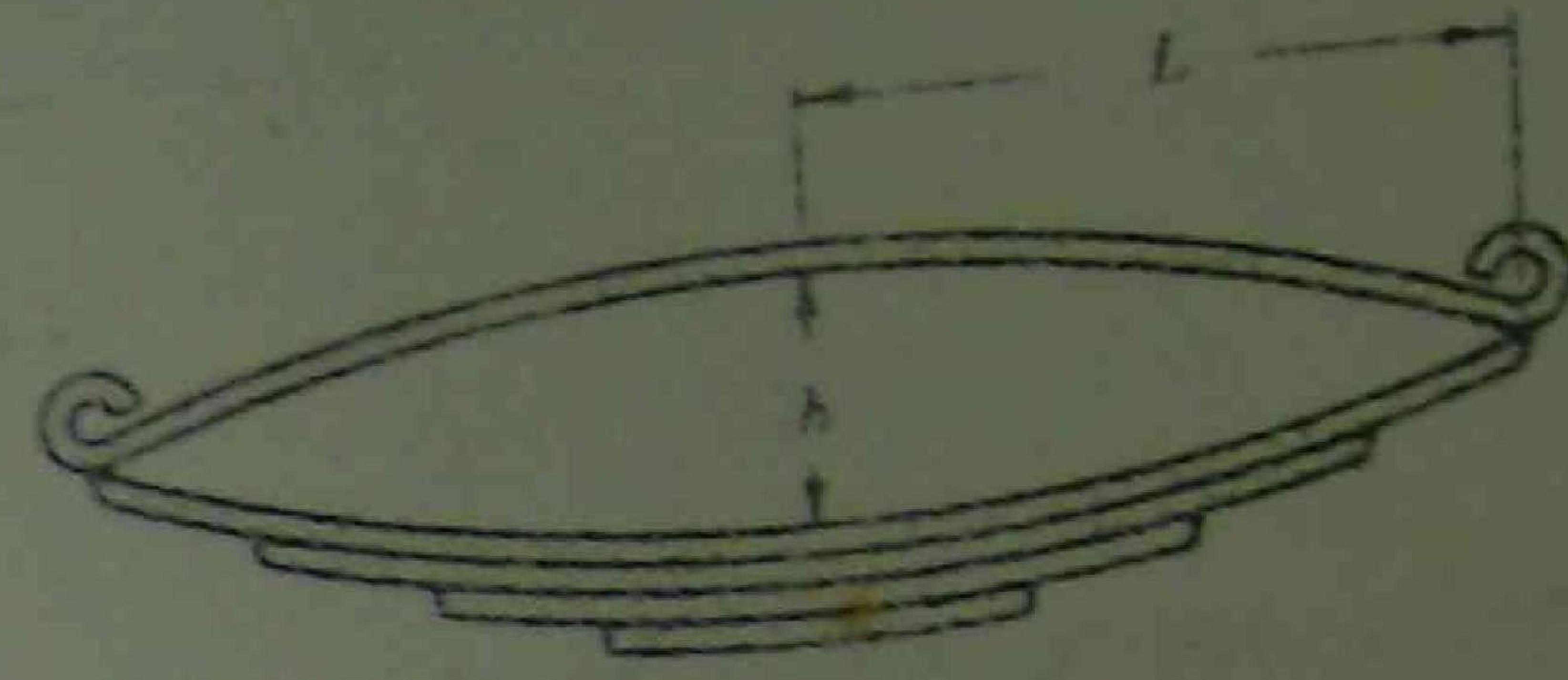


Fig. 16-4

number of full length leaves, n. The extra full length leaves are not beams of constant strength and will have a bending stress approximately 50% greater than the graduated le. es unless they are pre-stressed during assembly. Pre-stressing can be accomplished by having extra full length leaves formed with a different radius of curvature than the graduated leaves. This will leave a gap h between the extra full length leaves and the graduated leaves before assembly, as shown in Fig. 16-4 above. Then, upon assembly, the extra full length leaves will have an initial stress of opposite sign to that which occurs when the load is applied. The gap may be determined so that all of the leaves will be equally stressed after the full load F is applied.

THE BENDING STRESS, s_e, in the extra full length leaves if they are installed without an initial pre-stress will be

$$s_e = \frac{18\,FL}{bt^2(3n_e + 2n_g)}$$

where
- F = total applied load at the end of the spring, lb
- L = length of the cantilever or half the length of the semi-elliptic spring, in.
- b = width of each spring leaf, in.
- t = thickness of each spring leaf, in.
- n_e = number of extra full length leaves
- n_g = number of graduated leaves.

THE BENDING STRESS, s_g, in the graduated leaves if they are assembled with extra full length leaves without an initial pre-stress will be

$$s_g = \frac{12\,FL}{bt^2(3n_e + 2n_g)} = \frac{2s_e}{3}$$

THE DEFLECTION OF A MULTI-LEAF SPRING composed of graduated and extra full length leaves will be

$$y = \frac{12\,FL^3}{bt^3E(3n_e + 2n_g)}$$

where y = deflection at the end of the spring, in., E = modulus of elasticity, psi.

This equation will determine the deflection if $n_e = 0$, and also if the extra full length leaves are pre-stressed or if they are not pre-stressed.

THE BENDING STRESS, s, in multi-leaf springs without extra full length leaves or with extra full length leaves which have been pre-stressed so that all of the leaves have the same stress after the full load has been applied can be determined by

$$s = \frac{6FL}{nbt^2}$$

where s = bending stress, psi, and n = total number of leaves.

THE BENDING STRESS will be the same in all of the leaves of a multi-leaf spring composed of graduated and extra full length leaves if the extra full length leaves are pre-stressed by having the leaves pre-formed so that the gap h as shown in Fig. 16-4 above is

$$h = \frac{2FL^3}{nbt^3E}$$

where h = the gap between pre-assembled graduated leaves and extra full length leaves, in

HELICAL SPRINGS are usually made of circular cross section wire or rod as shown in Fig. 16-5. These springs are subjected to a torsional shear stress and to a transverse shear stress. There is also an additional stress effect due to the curvature of the helix. In order to take into account the effects of transverse shear and curvature, it is customary to multiply the torsional shear stress by a correction factor K, called the Wahl factor.

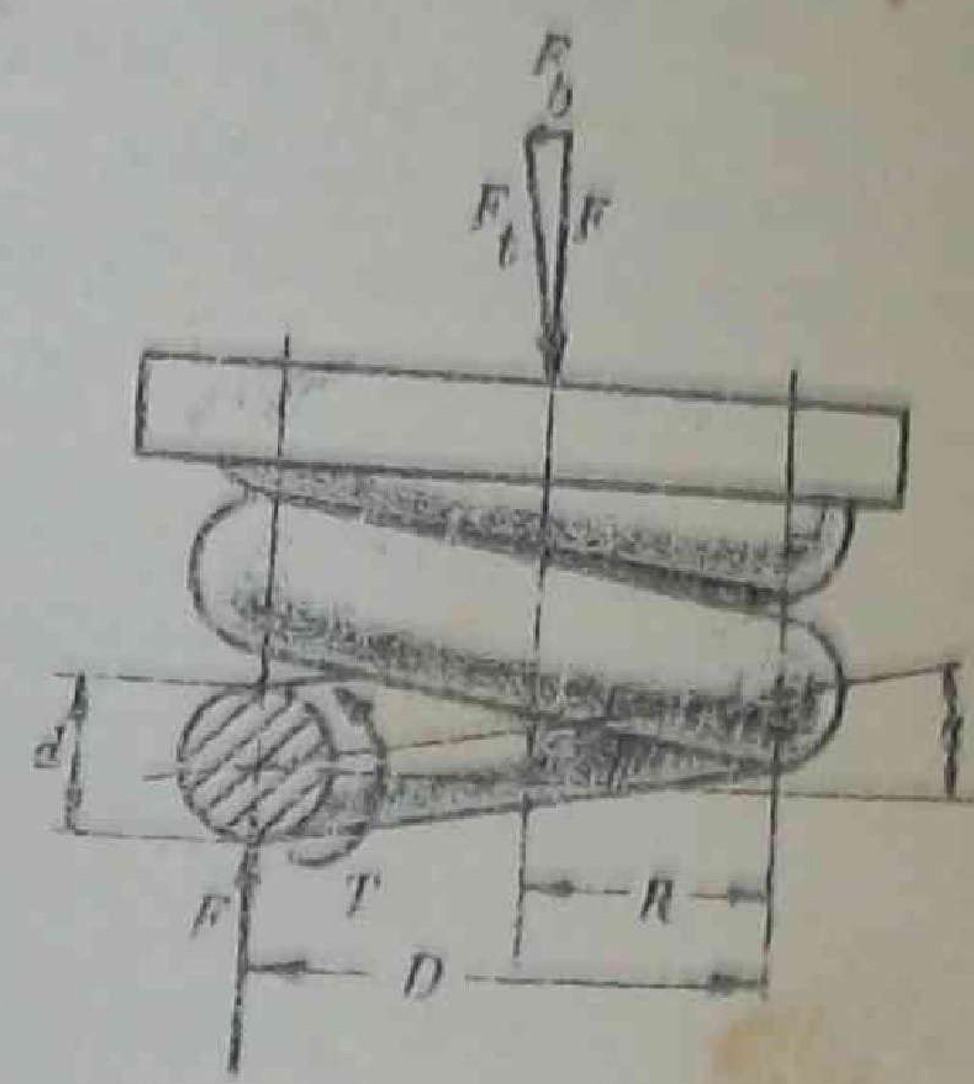


Fig. 16-5

THE SHEAR STRESS induced in a helical spring due to an axial load F is

$$s_s = K\frac{8FD}{\pi d^3} = K\frac{8FC}{\pi d^2}$$

where
s_s = total shear stress, psi;
D = mean diameter of coil, in.;
$K = \frac{4C-1}{4C-4} + \frac{0.615}{C}$, called the Wahl factor;
F = axial load, lb;
d = diameter of wire, in.;
$C = \frac{D}{d}$, called the spring index.

THE DEFLECTION of a helical spring due to an axial load F is

$$y = \frac{8FD^3n}{d^4G} = \frac{8FC^3n}{dG}$$

where n = number of active coils; y = axial deflection, in.; G = modulus of rigidity, psi.

THE SPRING RATE, or spring constant, is defined as the pounds per inch of deflection.

$$k = \frac{F}{y}$$

$k = \frac{Gd}{8C^3n}$ for a helical spring under axial load

THE SPRING RATE for springs in parallel having individual spring rates as shown in Fig. 16-6(a) below is

$$k = k_1 + k_2 + k_3$$

THE SPRING RATE for springs in series as shown in Fig. 16-6(b) below is

$$k = \frac{1}{1/k_1 + 1/k_2 + 1/k_3}$$

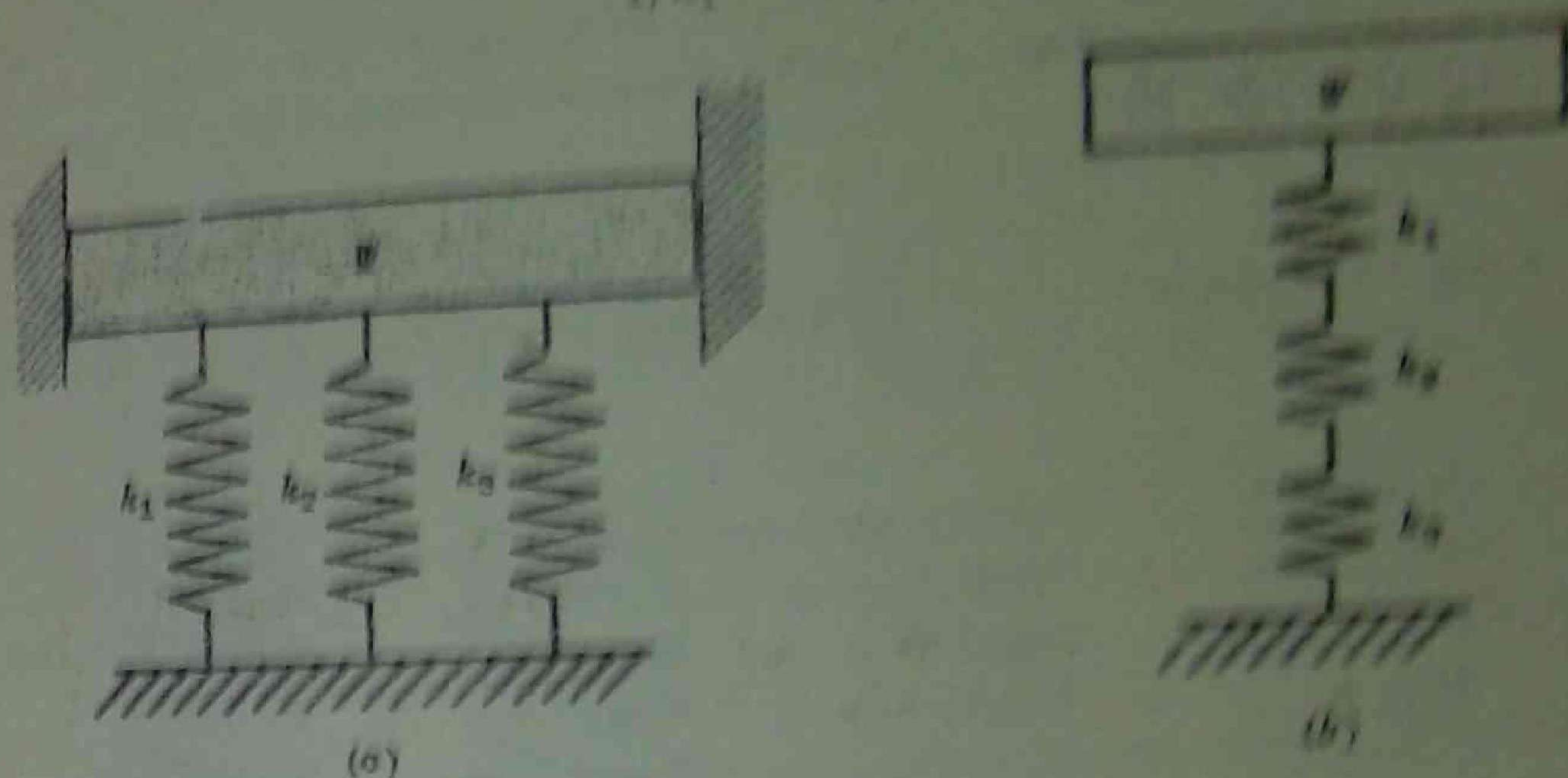


Fig. 16-6

THE ENERGY STORED, $(Eng)_s$, in springs having a linear force deflection relationship and obeying Hooke's law, can be determined by

$$(Eng)_s = \tfrac{1}{2}Fy \qquad \text{or} \qquad (Eng)_s = \tfrac{1}{2}T\theta$$

For a helical spring subjected to an axial load F, the energy stored is

$$(Eng)_s = \frac{s_s^2}{4G} \text{ in-lb per in}^3$$

For a helical spring subjected to a torsional load, the energy stored is

$$(Eng)_s = \frac{s^2}{8E} \text{ in-lb per in}^3 \quad \text{(for round wire)}$$

$$(Eng)_s = \frac{s^2}{6E} \text{ in-lb per in}^3 \quad \text{(for rectangular wire)}$$

For a cantilever beam of constant strength subjected to a bending force at the end, the amount of energy stored is

$$(Eng)_s = \frac{s^2}{6E} \text{ in-lb per in}^3$$

For a spiral spring subjected to a torsional load, the energy stored is

$$(Eng)_s = \frac{s^2}{6E} \text{ in-lb per in}^3$$

where

s_s = shear stress, psi
s = bending stress, psi
T = torque, in-lb
E = modulus of elasticity, psi
G = modulus of rigidity, psi
y = linear deflection, in.
θ = angular deflection, rad.

SPRING ENDS for helical springs may be either plain, plain ground, squared, or squared and ground as shown in Fig. 16-7 below. This results in a decrease of the number of active coils and affects the free length and solid length of the spring as shown below.

Type of Ends	Total Coils	Solid Length	Free Length
Plain	n	$(n+1)d$	$np+d$
Plain ground	n	nd	np
Squared	$n+2$	$(n+3)d$	$np+3d$
Squared and ground	$n+2$	$(n+2)d$	$np+2d$

p = pitch, n = number of active coils, d = wire diameter

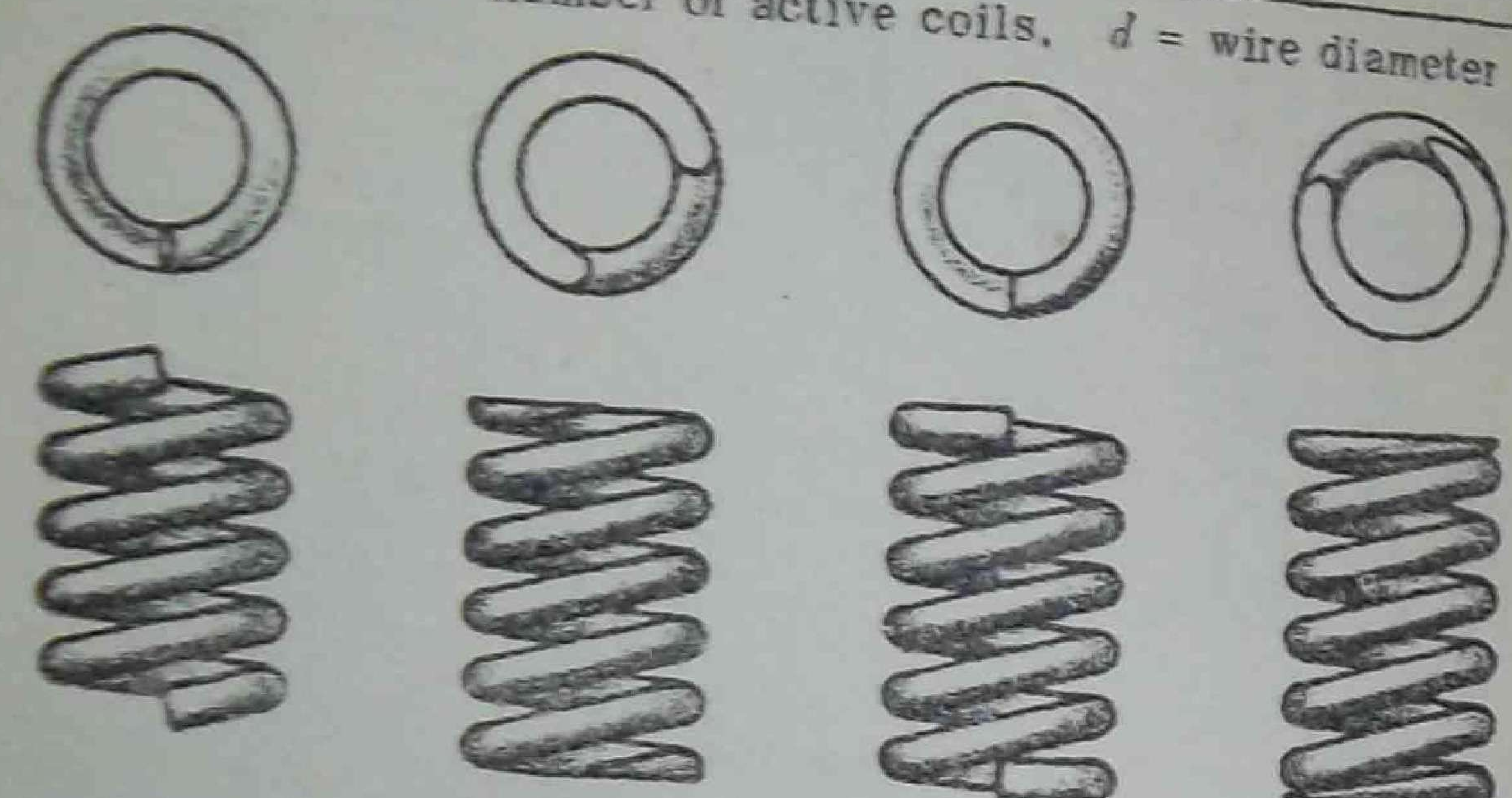


Fig. 16-7

BUCKLING MAY OCCUR IN COMPRESSION SPRINGS if the free length is over 4 times the mean diameter unless the spring is properly guided. The critical axial load that will cause buckling may be approximated by

$$F_{cr} = kL_f K_L$$

where
F_{cr} = axial load to produce buckling, lb
k = spring rate, lb/in. of axial deflection
L_f = free length of the spring, in.
K_L = a factor depending on the ratio L_f/D.

Hinged Ends		Built-in Ends	
L_f/D	K_L	L_f/D	K_L
1	0.72	1	0.72
2	0.63	2	0.71
3	0.38	3	0.68
4	0.20	4	0.63
5	0.11	5	0.53
6	0.07	6	0.38
7	0.05	7	0.26
8	0.04	8	0.19

SURGING may occur in helical springs which have loads applied repetitively at a rate close to the natural frequency of the spring. To avoid this possibility it is advisable that the natural frequency of the spring be at least 20 times the frequency of the applied load. The natural frequency of a steel coil, f_n, in cycles per minute can be determined by

$$f_n = \frac{761{,}500\,d}{nD^2}$$

ALLOWABLE STRESSES FOR HELICAL SPRINGS SUBJECTED TO STATIC LOADING may be based on the elastic limit in torsion. For static loading a factor of safety of 1.5 has been recommended to be applied to the torsional yield strength of the material. In determining the maximum stress induced in the spring, one method is to apply the portion of the Wahl factor that corrects for the transverse shear effect, but not the portion that corrects for curvature, since the latter has the nature of a stress concentration and is not serious in ductile materials subjected to static loads.

In the Wahl factor, $K = \frac{4C-1}{4C-4} + \frac{0.615}{C}$, the $\frac{4C-1}{4C-4}$ corrects for curvature and the $\frac{0.615}{C}$ corrects for transverse shear. The use of the full value of K for static loading would result in a conservative design. The Wahl factor may be considered as composed of two sub factors, K_s and K_c. The K_s shear stress sub-factor to be applied to the mean stress may be determined, according to Wahl, by

$$K_s = 1 + 0.5/C$$

which is based on a uniform transverse shear stress distribution. Then the design equations are

$$\frac{s_{ys}}{1.5} = K_s \frac{8FD}{\pi d^3} \quad \text{or} \quad \frac{s_{ys}}{1.5} = K \frac{8FD}{\pi d^3} \quad \text{for more conservative design}$$

ALLOWABLE STRESSES FOR HELICAL SPRINGS SUBJECTED TO FATIGUE LOADING are based on the endurance strength of the material. Wahl suggests two methods which are essentially as follows. In the first method use is made of a modified Soderberg line. Spring materials are usually tested for torsional endurance strength under a repeated stress that varies from zero to a maximum. Since test data for this released loading is available and since springs are ordinarily loaded in one direction only, more accurate results can be obtained by using the modified Soderberg diagram, as shown in Figure 16-8; the endurance limit for released loading, s_{rel}, is shown at a point A (where the mean stress is equal to $\frac{1}{2}s_{rel}$ and the variable stress is also equal to $\frac{1}{2}s_{rel}$). A line drawn from point A to point B, the yield point in shear, will give the failure line for fatigue and will correspond to test data better than the line from the endurance strength in reversed shear to the yield point in shear. The design line CD is drawn parallel to the line AB, with point D being located at s_{ys}/N. The mean stress s_m is plotted as the abscissa and the variable stress s_v is plotted as the ordinate.

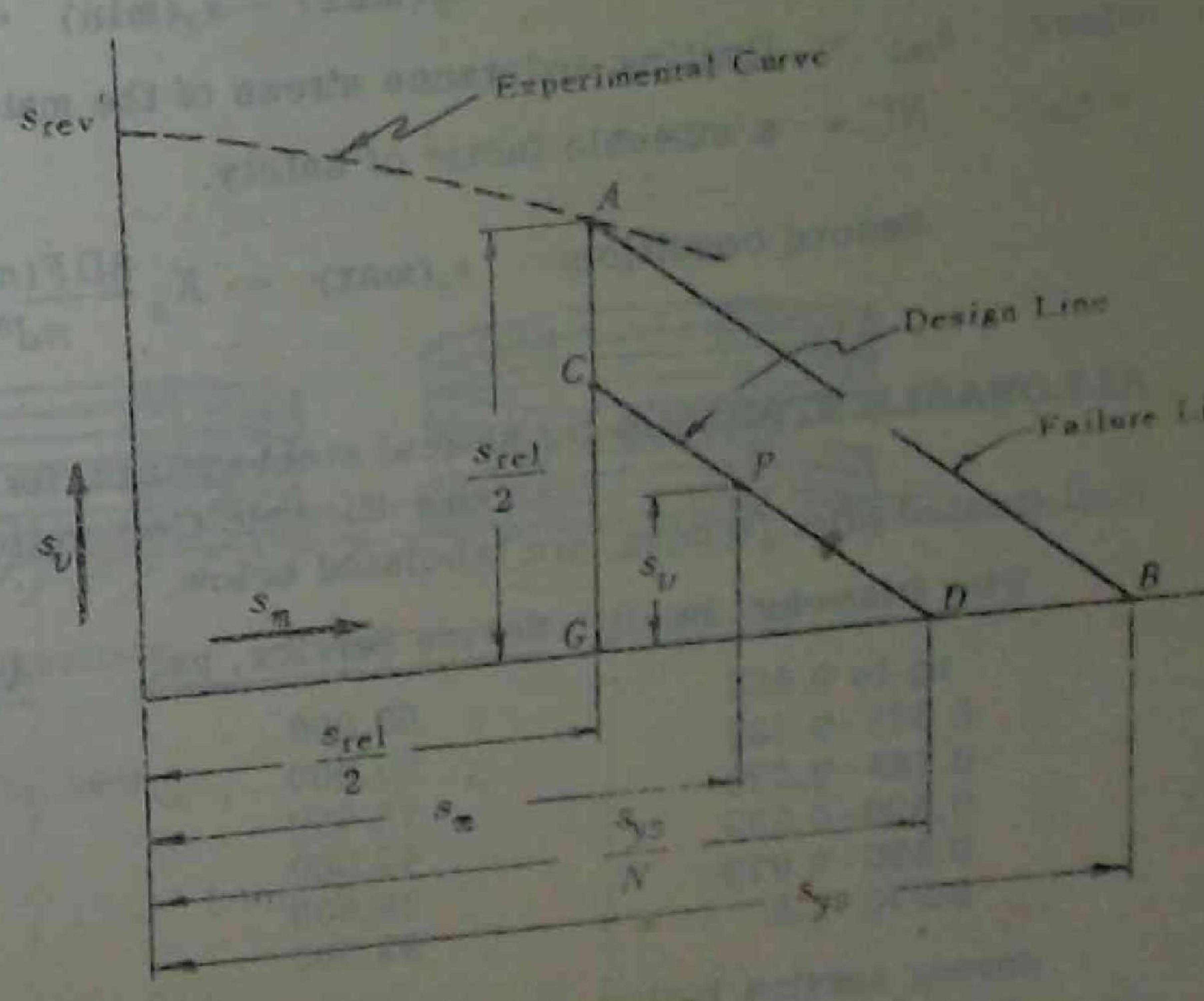


Fig. 16-8

The variable stress s_v may be calculated using the full value of the Wahl factor, $K = K_c K_s$, or the K_c may be reduced if information is available regarding the sensitivity of the material to this stress concentration effect due to curvature. Some materials are less sensitive than others.

$$s_v = K \frac{8F_v D}{\pi d^3} \qquad \text{where} \quad F_v = \frac{F(\max) - F(\min)}{2}$$

The average stress s_m may be calculated using only the static portion of the Wahl factor, K_s, which agrees with experimental evidence.

$$s_m = K_s \frac{8F_m D}{\pi d^3} \qquad \text{where} \quad F_m = \frac{F(\max) + F(\min)}{2}$$

Referring to the fatigue-stress diagram, line AB is the failure line. Hence we may construct line CD parallel to and below line AB to allow for a reasonable factor of safety N based on the yield shear strength. An equation may be written for line CD. With the origin at G, the coordinates of P are $[(s_m - \frac{1}{2}s_{rel}), s_v]$; the slope of CD = slope of $AB = -\dfrac{\frac{1}{2}s_{rel}}{s_{ys} - \frac{1}{2}s_{rel}}$; and intercept $CG = AG(\dfrac{CD}{GB}) = \frac{1}{2}s_{rel}\left(\dfrac{s_{ys}/N - \frac{1}{2}s_{rel}}{s_{ys} - \frac{1}{2}s_{rel}}\right)$. Then

$$s_v = -\left(\frac{\frac{1}{2}s_{rel}}{s_{ys} - \frac{1}{2}s_{rel}}\right)(s_m - \tfrac{1}{2}s_{rel}) + \tfrac{1}{2}s_{rel}\left(\frac{s_{ys}/N - \frac{1}{2}s_{rel}}{s_{ys} - \frac{1}{2}s_{rel}}\right)$$

from which

$$N = \frac{s_{ys}}{s_m - s_v + 2s_v s_{ys}/s_{rel}}$$

This may be used as a design equation, since all points on line CD represent a combination of variable and mean stress conditions which are safe. The factor of safety may be taken as 1.8, more or less, depending upon operating conditions. Values for s_{rel} are not very complete, but based on various current sources a value of $36{,}000/d^{0.2}$ seems to be a good approximation for oil-tempered carbon steel for wire diameters up to 0.625 in.

In the second method suggested by Wahl, one condition is that the stress range is calculated using the full value of the Wahl factor K. Also, a second condition is that the peak stress calculated using only K_s must not exceed the yield strength of the material divided by a suitable factor of safety.

First condition: $\quad s_s(\max) - s_s(\min) = K \dfrac{8D[F(\max) - F(\min)]}{\pi d^3} = \dfrac{s_{rel}}{N}$

where $\quad s_{rel}$ = limiting endurance stress of the material

N = a suitable factor of safety.

Second condition: $\quad s_s(\max) = K_s \dfrac{8DF(\max)}{\pi d^3} = \dfrac{s_{ys}}{N}$

ALLOWABLE STRESSES for helical steel springs, for one specific material, as published by the Westinghouse Electric Corporation for SAE 6150 oil-tempered hot wound springs heat-treated after forming, are tabulated below.

Wire Diameter, in.	Severe Service, psi	Average Service, psi	Light Service, psi
up to 0.085	60,000	75,000	93,000
0.085-0.185	55,000	69,000	85,000
0.185-0.320	48,000	60,000	74,000
0.320-0.530	42,000	52,000	65,000
0.530-0.970	36,000	45,000	56,000
0.970-1.5	32,000	40,000	50,000

Severe service includes rapid continuous loading where the ratio of minimum to maximum stress is one-half or less. Average service is the same as severe except for intermittent operation. Light service includes springs subjected to static loads or to infrequently varied loads.

BELLEVILLE SPRINGS are made up from tapered washers as shown in Fig. 16-9(*a*) below. The washers may be stacked in series, parallel, or a combination of parallel-series as shown in Fig. 16-9(*b*) below. The load-deflection and stress-deflection formulas for one washer as given by Almen and Laszlo (ASME transactions, May 1936, Volume 58, No. 4) are:

$$P = \frac{Ey}{(1-\mu^2)M(d_o/2)^2}\left[(h - y/2)(h - y)t + t^3\right]$$

$$s = \frac{Ey}{(1-\mu^2)M(d_o/2)^2}\left[C_1(h - y/2) + C_2 t\right]$$

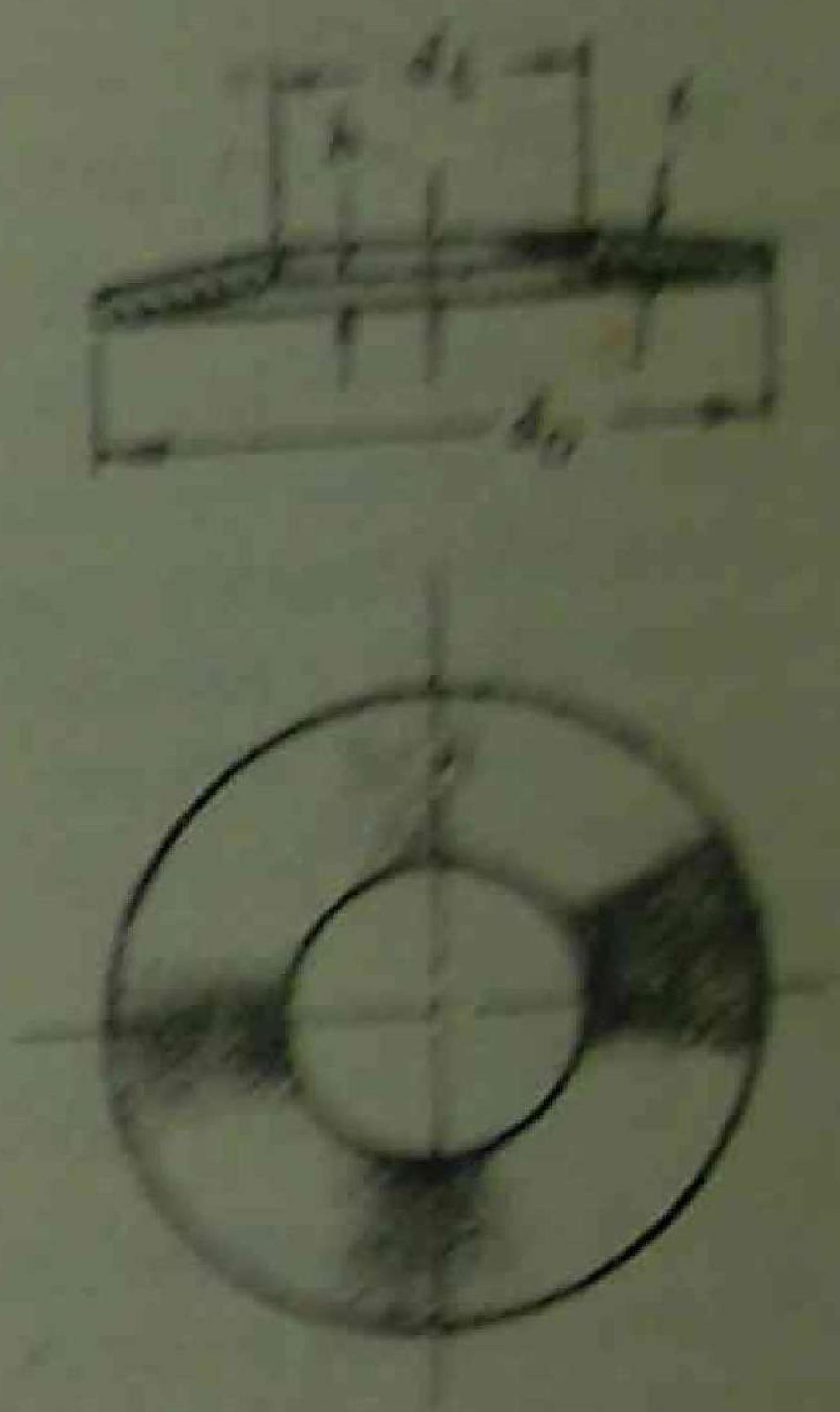


Fig. 16-9(*a*)

where

P = axial load, lb

y = deflection, in.

t = thickness of washer, in.

h = free height minus thickness, in.

E = modulus of elasticity, psi

s = stress at inside circumference, psi

d_o = outside diameter of washer, in.

d_i = inside diameter of washer, in.

μ = Poisson's ratio (0.3 for steel)

$$M = \frac{6}{\pi \log_e(d_o/d_i)}\left[\frac{d_o/d_i - 1}{d_o/d_i}\right]^2$$

$$C_1 = \frac{6}{\pi \log_e(d_o/d_i)}\left[\frac{d_o/d_i - 1}{\log_e(d_o/d_i)} - 1\right]$$

$$C_2 = \frac{6}{\pi \log_e(d_o/d_i)}\left[\frac{d_o/d_i - 1}{2}\right]$$

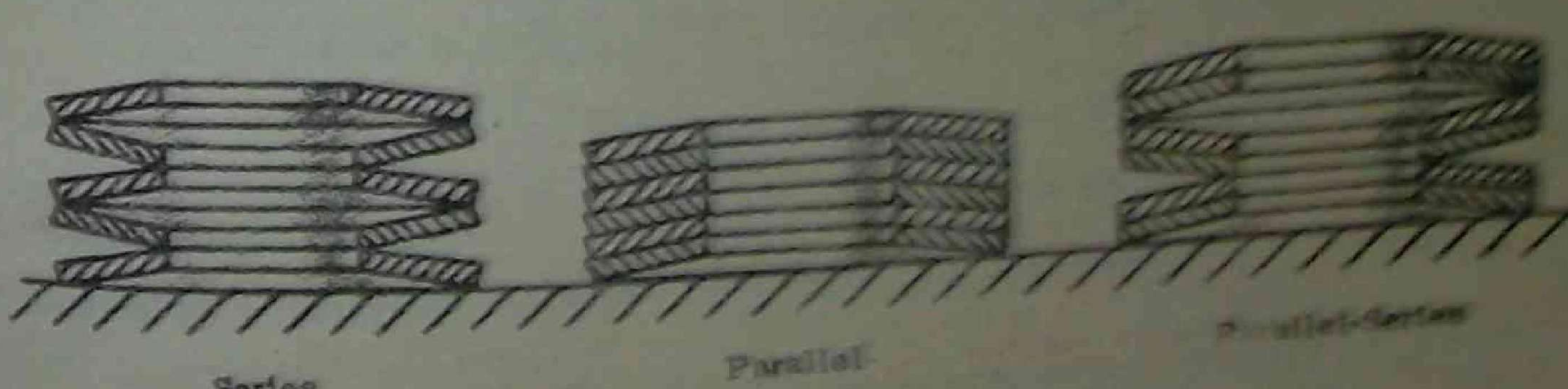


Fig. 16-9(*b*)

SOLVED PROBLEMS

1. Derive the stress, force, and deflection relationships for multi-leaf springs.

Solution:

Consider a cantilever beam of constant strength and uniform thickness t, as shown in Fig. 16-2(a), to be cut into n strips of width b and stacked in a graduated manner as shown in Fig. 16-2(b).

The bending stress is the same at all sections of the triangular beam. We will assume that this situation prevails after the strips are stacked, even though this is not entirely true.

$$s = \frac{Mc}{I} = \frac{FL(\frac{1}{2}t)12}{nbt^3} = \frac{6FL}{nbt^2}$$

The deflection of a beam of constant strength and uniform thickness is

$$y = \frac{FL^3}{2EI_{(max)}} = \frac{6FL^3}{Ebnt^3}$$

These equations also apply to the semi-elliptic leaf spring, which may be considered as two cantilevers supported at its center as shown in Fig. 16-3.

The addition of one or more extra full length leaves, n_e, of constant width and thickness on top of the graduated stack is approximately equivalent to having beam e of constant width loaded in parallel with beam g of constant strength, as shown in Fig. 16-10. The deflections of beams e and g are

$$y_e = \frac{F_e L^3}{3EI_{(max)e}} \quad \text{and} \quad y_g = \frac{F_g L^3}{2EI_{(max)g}}$$

where F_e and F_g represent the portions of the total force F absorbed by beams e and g. Since the deflections are equal, we may equate $y_e = y_g$ or

$$\frac{F_e L^3}{3EI_{(max)e}} = \frac{F_g L^3}{2EI_{(max)g}}$$

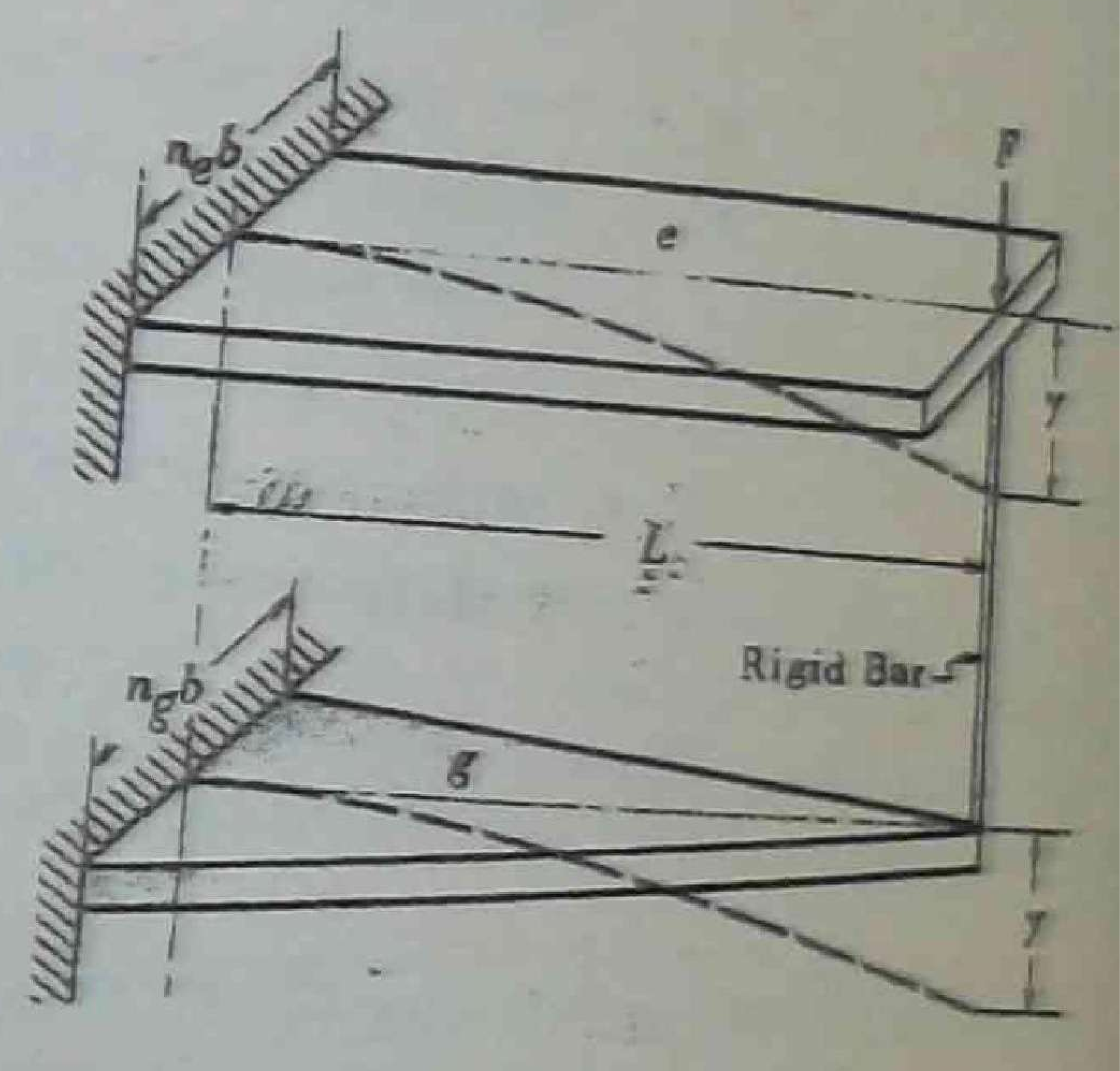


Fig. 16-10

Let n_g and n_e equal the number of graduated leaves and extra full length leaves respectively. Then $I_{(max)e} = n_e bt^3/12$ and $I_{(max)g} = n_g bt^3/12$. Substituting these values in the previous equation, $\dfrac{F_e}{3n_e} = \dfrac{F_g}{2n_g}$ or $F_e = \dfrac{3n_eF_g}{2n_g}$. Now

$$F = F_e + F_g, \qquad F_e = \frac{3n_e}{3n_e + 2n_g}F$$

$$s_e = \frac{6F_eL}{n_ebt^2} = \frac{18FL}{bt^2(3n_e + 2n_g)}, \qquad s_g = \frac{6F_gL}{n_gbt^2} = \frac{12FL}{bt^2(3n_e + 2n_g)}$$

The deflection of the composite spring is $y = \dfrac{12FL^3}{bt^3E(3n_e + 2n_g)}$

2. Derive the stress, force, and deflection relationship for a helical coil spring for a concentric axial load.

Solution:

Referring to Fig. 16-5, the F_y component of the axial force F produces a bending stress s,

$$s = \frac{16FD\sin\alpha}{\pi d^3} \quad \text{(neglecting curvature effects)}$$

This stress may be neglected for small helix angles α. The axial force F produces torsional stress s_s. Since $T = \frac{1}{2}FD\cos\alpha \approx \frac{1}{2}FD$ for small helix angles,

$$s_s = \frac{TR}{J} = \frac{8FD}{\pi d^3}$$

where F = axial load, lb; D = mean diameter of the coil, in.; d = diameter of the wire, in.; y = axial deflection, in.; s_s = shearing stress, psi.

In addition to the torsional shear stress, there is a transverse shear and an additional stress due to the curvature of the coil. In order to include both of these effects, a stress factor K, called the Wahl factor, may be used.

$$s_s = K\frac{8FD}{\pi d^3}$$

where $K = \dfrac{4C-1}{4C-4} + \dfrac{0.615}{C}$ and the spring index $C = D/d$.

An equation for the deflection of a helical spring may be obtained by equating the work required to deflect the spring to the torsional energy absorbed by the twisted wire. The helical spring having n active coils is developed into a straight rod of diameter d and length $n\pi D/(\cos\alpha)$, as shown in Fig. 16-11. Cos α may be taken as unity since the helix angle is usually small. Then

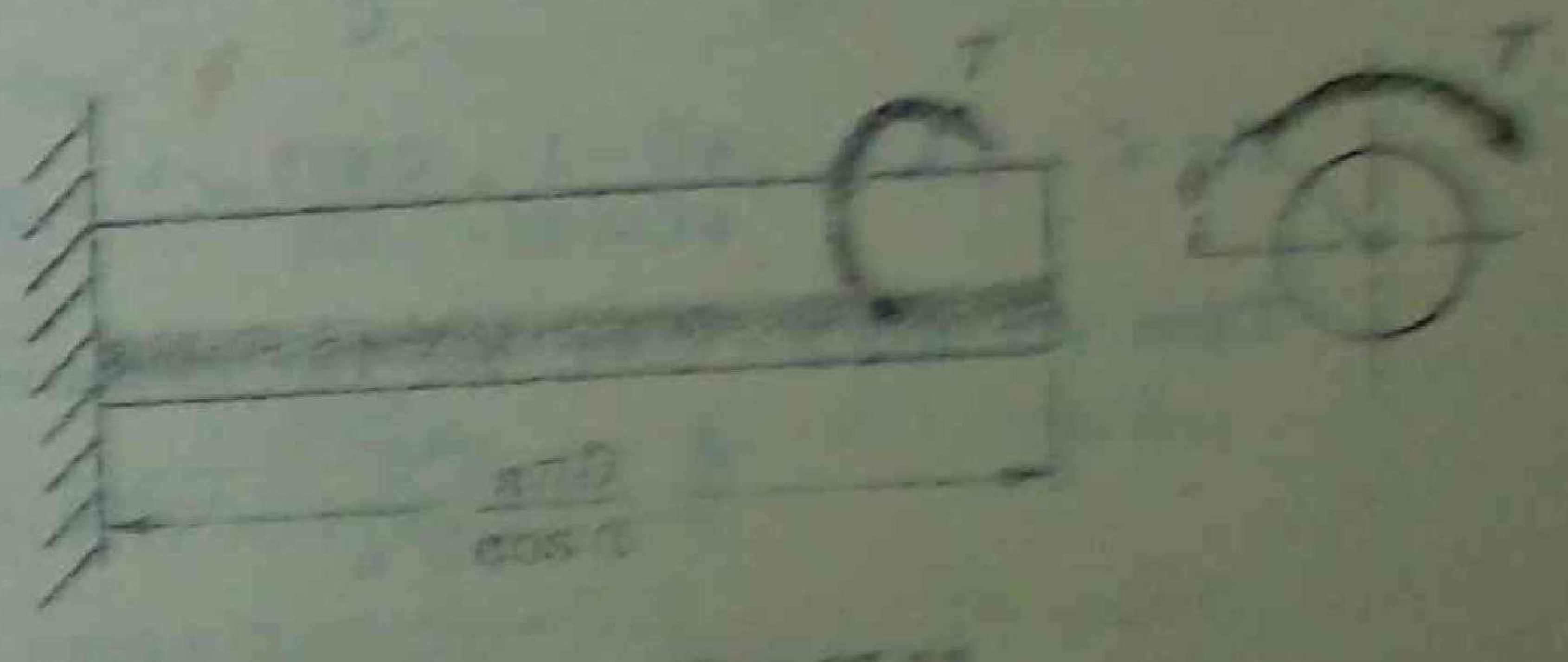


Fig. 16-11

Work in = energy absorbed

$$\tfrac{1}{2}Fy = \tfrac{1}{2}T\theta = \tfrac{1}{2}(\tfrac{1}{2}FD)\theta \quad \text{or} \quad y = \tfrac{1}{2}D\theta$$

Since $\theta = \dfrac{Tn\pi D}{JG} = \dfrac{16FD^2n}{d^4G}$, where G = torsional modulus of elasticity (psi),

$$y = \frac{8FD^3n}{d^4G} = \frac{8FC^3n}{dG}$$

The value of G for spring steel is approximately 11.5×10^6 psi.

3. A 35 in. long cantilever spring is composed of 8 graduated leaves and one extra full length leaf. The leaves are $1\frac{3}{4}$ in. wide. A load of 500 lb at the end of the spring causes a deflection of 3 in. Determine the thickness of the leaves and the maximum bending stress in the full length leaf assuming first that the extra full length leaf has been pre-stressed to give the same stress in all the leaves, and then determine the stress in the full extra length leaf assuming no pre-stress.

Solution:

$$y = \frac{12FL^3}{bt^3(2n_g+3n_e)E}, \qquad 3 = \frac{(12)(500)(35)^3}{(1.75)(t^3)(16+3)(30\times10^6)}, \qquad t = 0.442 \text{ in.}$$

With extra full length leaf pre-stressed, $s = \dfrac{6FL}{nbt^2} = \dfrac{6(500)(35)}{9(1.75)(0.442)^2} = 34{,}100$ psi.

With no pre-stress, $s_e = \dfrac{18FL}{bt^2(2n_g+3n_e)} = \dfrac{(18)(500)(35)}{(1.75)(0.442)^2(16+3)} = 48{,}500$ psi.

4. Determine the required number of coils and permissible deflection in a helical spring made of 1/16 inch diameter steel wire, assuming a spring index of 6 and an allowable stress of 50,000 psi in shear. The spring rate is to be 10 lb/in.

Solution:

Spring index $C = \dfrac{D}{d}$, $6 = \dfrac{D}{1/16}$, $D = 0.375$ in. Wahl factor $K = \dfrac{4C-1}{4C-4} + \dfrac{0.615}{C} = 1.25$

$$\frac{F}{y} = \frac{dG}{8C^3n}, \qquad 10 = \frac{(1/16)(11.5\times10^6)}{8(6)^3n}, \qquad n = 41.6 \text{ coils}$$

Find force and deflection: $s_s = K\dfrac{8FD}{\pi d^3}$, $50{,}000 = 1.25\dfrac{8F(0.375)}{\pi(1/16)^3}$, $F = 10.2$ lb

The deflection should be limited to 1 in. for a spring rate of 10 lb/in.

Chapter 19

Helical Gears

HELICAL GEARS differ from spur gears in that they have teeth that are cut in the form of a helix on their pitch cylinders instead of parallel to the axis of rotation. Helical gears may be used to connect either parallel or non-parallel shafts. The discussion in this chapter will be limited to helical gears connecting parallel shafts. In this case a right hand helix will always mesh with a left hand helix. A helical gear with a left hand helix is shown in Fig.19-1 below.

ψ = helix angle, degrees
F = transmitted force (torque producing force), lb
F_e = end thrust = $F \tan\psi$, lb
P_c = circumferential circular pitch, in.
P_{nc} = normal circular pitch, in.
b = face width, in.
P_d = diametral pitch in plane of rotation
P_{nd} = normal diametral pitch in plane normal to tooth

Note that: $P_{nc} = P_c \cos\psi$, $P_{nd} = P_d/\cos\psi$, $P_{nc} \times P_{nd} = \pi = P_c P_d$

In order that contact on the face of the tooth shall always contain at least one point on the pitch line, the minimum face width of the tooth is

$$b_{min} = \frac{P_c}{\tan\psi}$$

THE PRESSURE ANGLE ϕ_n in the normal plane is distinguished from the pressure angle ϕ in the transverse plane as shown in Fig.19-2 below, the relationship being

$$\tan\phi_n = \tan\phi \cos\psi$$

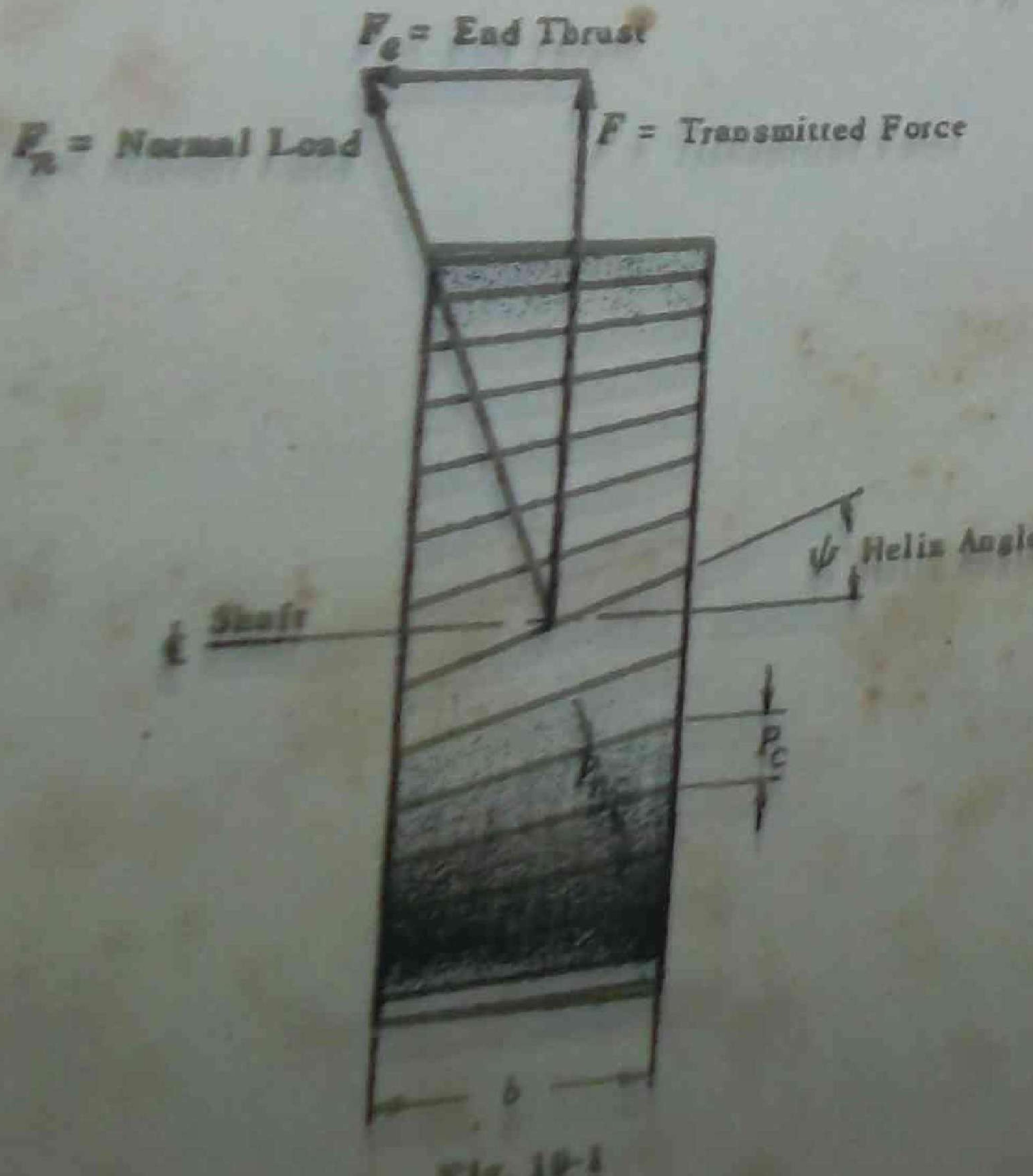


Fig. 19-1

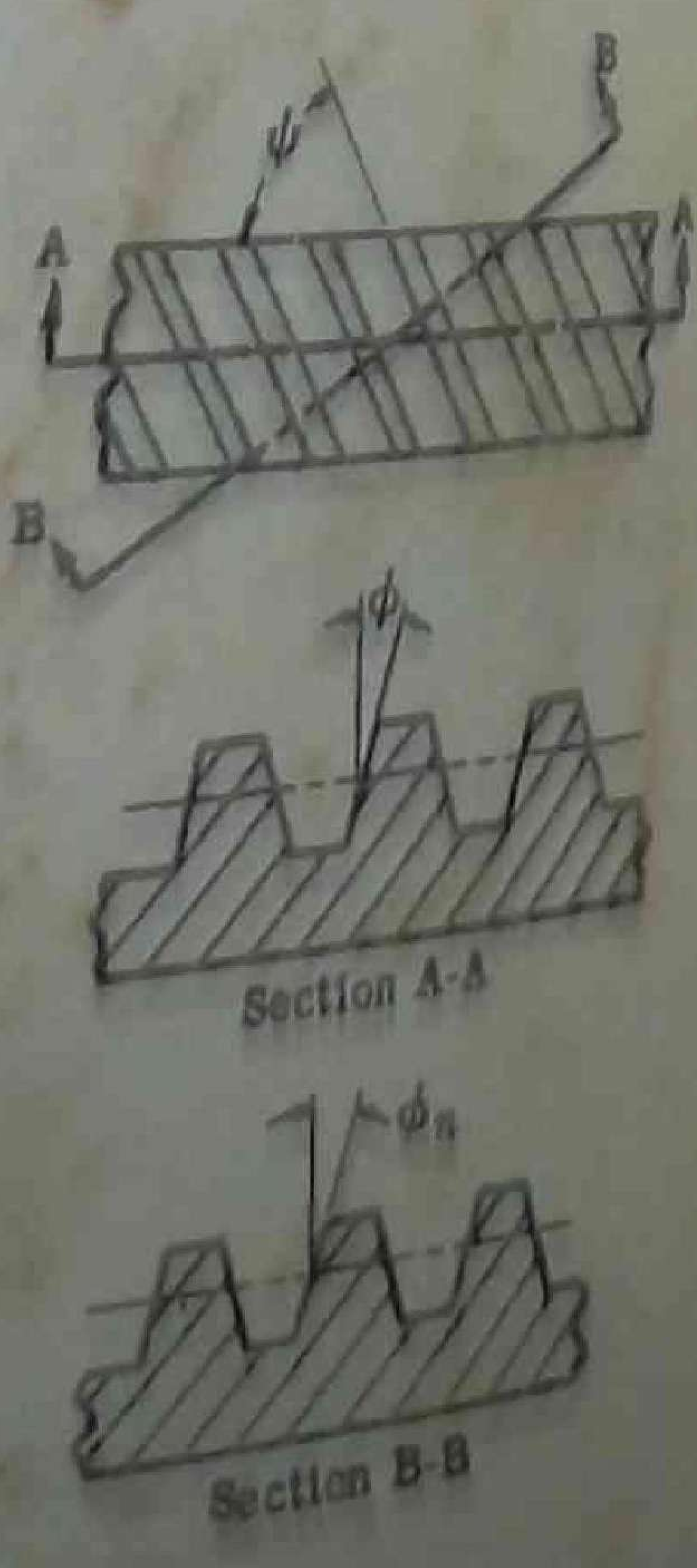


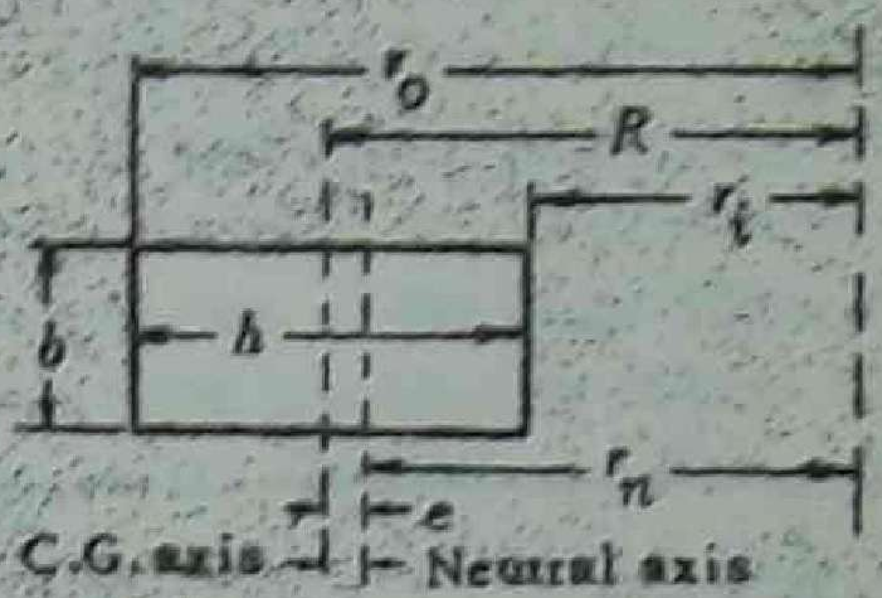
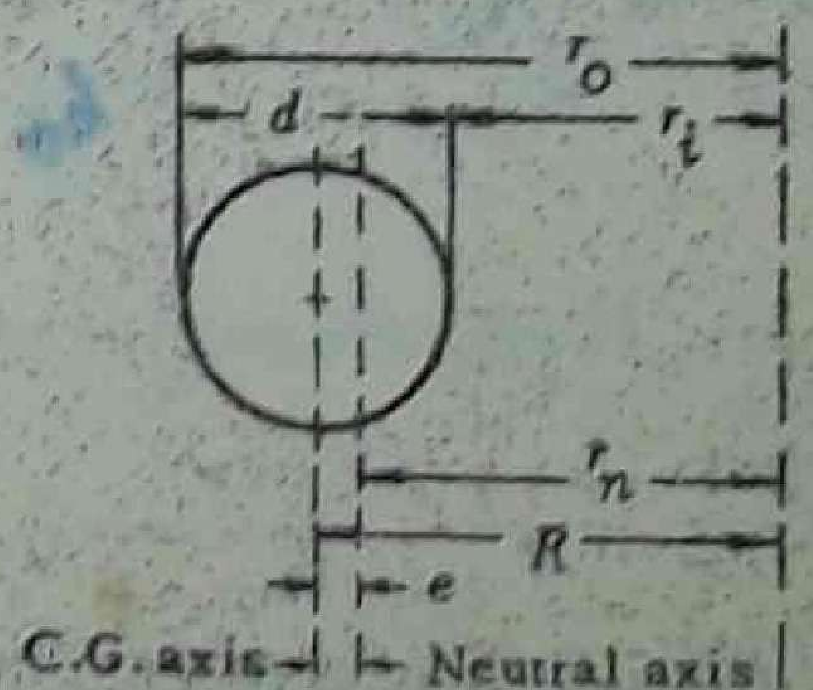
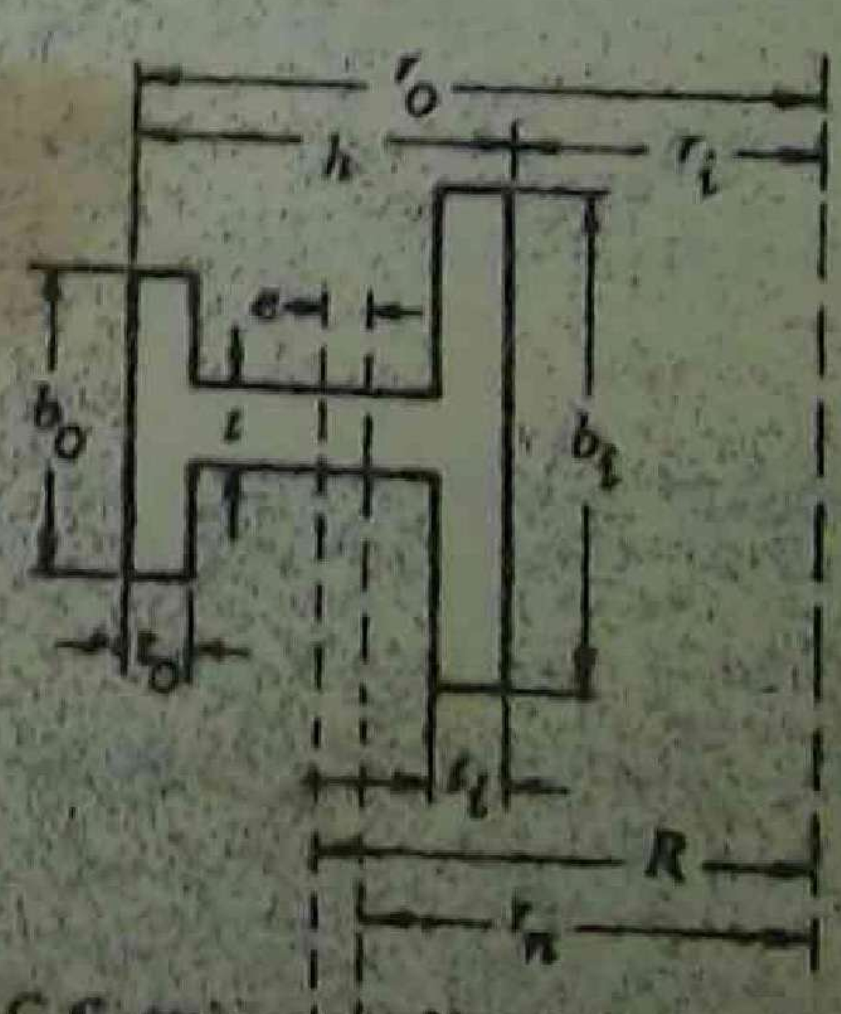
Fig. 19-2

If the section is a symmetrical one (as a circle, rectangle, I-beam with equal flanges) the maximum bending stress will always occur at the inside fiber. If the section is unsymmetrical, the maximum bending stress may occur at either the inside or outside fiber.

If the section has an axial load in addition to bending, the axial stress must be added algebraically to the bending stress.

Considerable care must be taken in the arithmetic. The distance "e" from the center of gravity axis to the neutral axis is usually small. A numerical variation in the calculation of "e" can cause a large percentage change in the final results.

Table I below gives the location to the neutral axis, the distance from the centroidal axis to the neutral axis, and the distance to the centroidal axis from the center of curvature for various commonly encountered shapes.

TABLE I	
	$r_n = \dfrac{h}{\log_e r_o/r_i}$ $e = R - r_n$ $R = r_i + h/2$
	$r_n = \dfrac{[r_o^{1/2} + r_i^{1/2}]^2}{4}$ $e = R - r_n$ $R = r_i + d/2$
	$r_n = \dfrac{(b_i - t)(t_i) + (b_o - t)(t_o) + th}{b_i \log_e \dfrac{r_i + t_i}{r_i} + t \log_e \dfrac{r_o - t_o}{r_i + t_i} + b_o \log_e \dfrac{r_o}{r_o - t_o}}$ $e = R - r_n$ $R = r_i + \dfrac{\frac{1}{2}h^2 t + \frac{1}{2}t_i^2(b_i - t) + (b_o - t)(t_o)(h - \frac{1}{2}t_o)}{(b_i - t)(t_i) + (b_o - t)(t_o) + th}$

THE VIRTUAL OR FORMATIVE number of teeth, N_f, on a helical gear is defined as the number of teeth that could be generated on the surface of a cylinder having a radius equal to the radius of curvature at a point at the tip of the minor axis of an ellipse obtained by taking a section of the gear in the normal plane.

$$N_f = \frac{N}{\cos^3 \psi}$$

where N = actual number of teeth and ψ = helix angle.

STRENGTH DESIGN for helical gears may be handled by employing design methods similar to those used for spur gears. Assuming that the load is distributed as for a spur gear, and looking at the tooth normal to the helix, the normal load F_n using the Lewis equation is

$$F_n = s\left(\frac{b}{\cos\psi}\right)\frac{\pi y}{P_{nd}}$$

Substituting the tangential force $F = F_n \cos\psi$, and $P_{nd} = P_d/\cos\psi$,

$$F = \frac{sby\pi}{P_{nd}} = \frac{sk\pi^2 y}{P_{nd}^2 \cos\psi} \qquad \text{(use where standard pitch is in normal plane)}$$

or

$$F = \frac{sby\pi\cos\psi}{P_d} = \frac{sk\pi^2 y\cos\psi}{P_d^2} \qquad \text{(use where standard pitch is in diametral plane)}$$

where $k = b/P_c$ (limited to a maximum of about 6)
P_d = diametral pitch in the plane of rotation
y = the form factor based on the virtual or formative number of teeth. If the pressure angle is standard in the normal plane, use y from the spur gear table. If the pressure angle in the diametral plane is standard, use y from the spur gear table as an approximation. A more accurate evaluation of y for the latter case could be obtained from a graphical solution.

The allowable stress s may be taken as approximately equal to the endurance limit of the material in released loading, corrected for stress concentration effects and multiplied by a velocity factor:

$$s = s_o\left(\frac{78}{78+\sqrt{V}}\right) \quad \text{allowable stress, psi}$$

where s_o = about one-third of the ultimate strength of the material, psi. This allows for an average stress concentration correction.
V = pitch line velocity, fpm.

A more accurate evaluation of s_o could be made if data regarding endurance limit and fatigue stress-concentration effects for the material are available. However, in view of other approximations, design based on the above expressions should, in general, be adequate since it must also be checked for dynamic load and wear load as explained later.

In the design check for strength, the pitch diameter is either known or unknown. If the pitch diameter is known, the following form of the Lewis equation may be used:

$$\frac{P_d^2}{y} = \frac{s_o k\pi^2\cos\psi}{F}\left(\frac{78}{78+\sqrt{V}}\right)$$

where $k = b/P_c$
F = tangential force = torque/(pitch radius), lb
V = pitch line velocity, fpm.

Then the above expression gives an *allowable* numerical value for the ratio P_d^2/y which controls the strength check.

If the pitch diameter is unknown, the following form of the Lewis equation may be used:

$$s = \frac{2TP_d^3}{ky\pi^2 N\cos\psi}$$

Chapter 19

Helical Gears

HELICAL GEARS differ from spur gears in that they have teeth that are cut in the form of a helix on their pitch cylinders instead of parallel to the axis of rotation. Helical gears may be used to connect either parallel or non-parallel shafts. The discussion in this chapter will be limited to helical gears connecting parallel shafts. In this case a right hand helix will always mesh with a left hand helix. A helical gear with a left hand helix is shown in Fig. 19-1 below.

ψ = helix angle, degrees
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P_c = circumferential circular pitch, in.
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P_d = diametral pitch in plane of rotation
P_{nd} = normal diametral pitch in plane normal to tooth

Note that: $P_{nc} = P_c \cos\psi$, $P_{nd} = P_d/\cos\psi$, $P_{nc} \times P_{nd} = \pi = P_c P_d$

In order that contact on the face of the tooth shall always contain at least one point on the pitch line, the minimum face width of the tooth is

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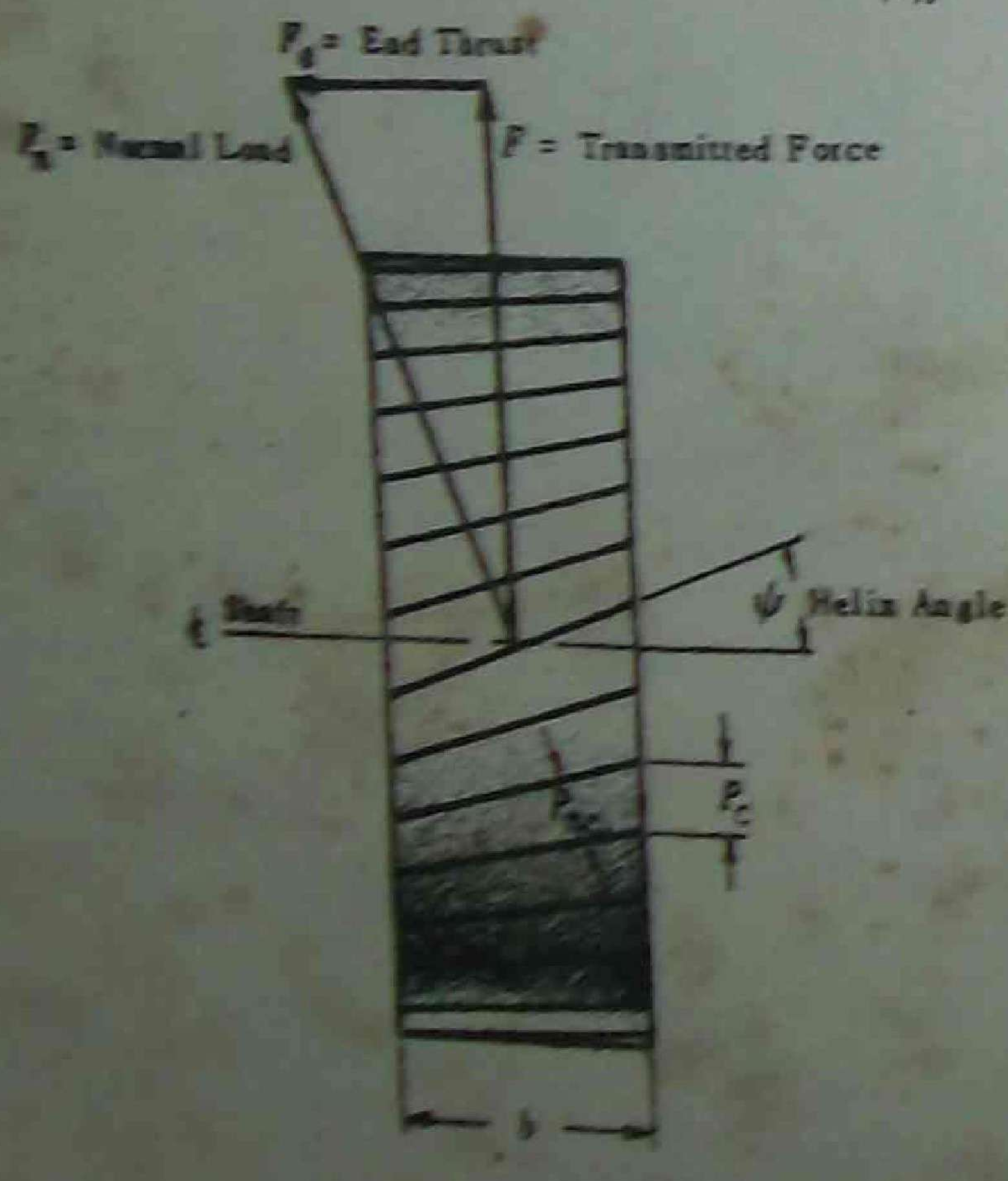


Fig. 19-1

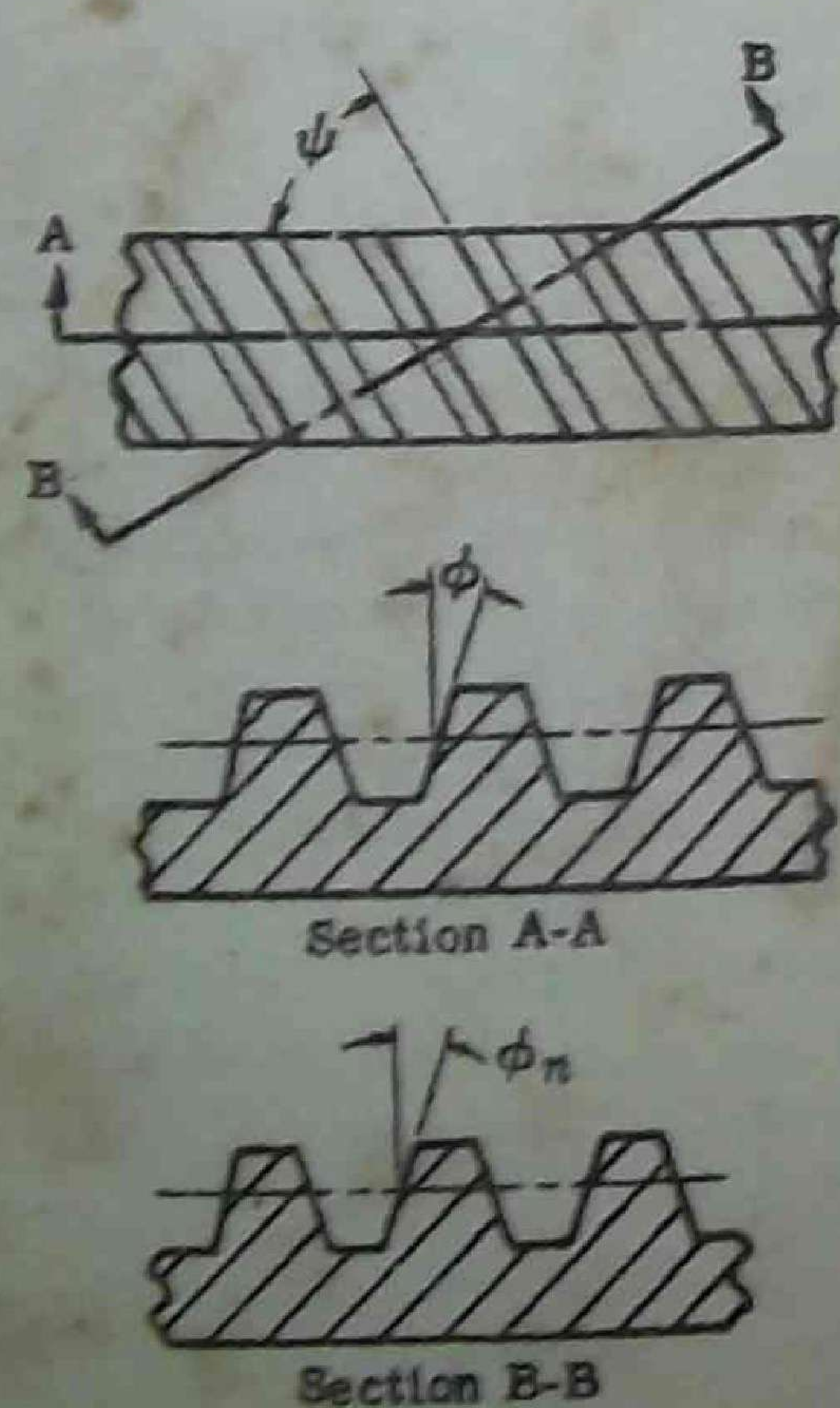


Fig. 19-2

THE VIRTUAL OR FORMATIVE number of teeth, N_f, on a helical gear is defined as the number of teeth that could be generated on the surface of a cylinder having a radius equal to the radius of curvature at a point at the tip of the minor axis of an ellipse obtained by taking a section of the gear in the normal plane.

$$N_f = \frac{N}{\cos^3 \psi}$$

where N = actual number of teeth and ψ = helix angle.

STRENGTH DESIGN for helical gears may be handled by employing design methods similar to those used for spur gears. Assuming that the load is distributed as for a spur gear, and looking at the tooth normal to the helix, the normal load F_n using the Lewis equation is

$$F_n = s\left(\frac{b}{\cos \psi}\right)\frac{\pi y}{P_{nd}}$$

Substituting the tangential force $F = F_n \cos \psi$, and $P_{nd} = P_d / \cos \psi$,

$$F = \frac{sby\pi}{P_{nd}} = \frac{sk\pi^2 y}{P_{nd}^2 \cos \psi} \qquad \text{(use where standard pitch is in normal plane)}$$

or

$$F = \frac{sby\pi \cos \psi}{P_d} = \frac{sk\pi^2 y \cos \psi}{P_d^2} \qquad \text{(use where standard pitch is in diametral plane)}$$

where $k = b/P_c$ (limited to a maximum of about 6)

P_d = diametral pitch in the plane of rotation

y = the form factor based on the virtual or formative number of teeth. If the pressure angle is standard in the normal plane, use y from the spur gear table. If the pressure angle in the diametral plane is standard, use y from the spur gear table as an approximation. A more accurate evaluation of y for the latter case could be obtained from a graphical solution.

The allowable stress s may be taken as approximately equal to the endurance limit of the material in released loading, corrected for stress concentration effects and multiplied by a velocity factor:

$$s = s_o\left(\frac{78}{78 + \sqrt{V}}\right) \quad \text{allowable stress, psi}$$

where s_o = about one-third of the ultimate strength of the material, psi. This allows for an average stress concentration correction.

V = pitch line velocity, fpm.

A more accurate evaluation of s_o could be made if data regarding endurance limit and fatigue stress-concentration effects for the material are available. However, in view of other approximations, design based on the above expressions should, in general, be adequate since it must also be checked for dynamic load and wear load as explained later.

In the design check for strength, the pitch diameter is either known or unknown. If the pitch diameter is known, the following form of the Lewis equation may be used:

$$\frac{P_d^2}{y} = \frac{s_o k \pi^2 \cos \psi}{F}\left(\frac{78}{78 + \sqrt{V}}\right)$$

where $k = b/P_c$

F = tangential force = torque/(pitch radius), lb

V = pitch line velocity, fpm.

Then the above expression gives an *allowable* numerical value for the ratio P_d^2/y which controls the strength check.

If the pitch diameter is unknown, the following form of the Lewis equation may be used:

$$s = \frac{2TP_d^3}{ky\pi^2 N \cos \psi}$$

where s = actual induced stress, psi
T = resisting torque of the weaker gear
N = actual number of teeth on the weaker gear

This expression gives a numerical value of the actual induced stress in terms of the diametral pitch.

The above procedures based on strength design should be considered only as a first approximation in arriving at a possible pitch and face width which must be checked for wear load and dynamic load.

THE LIMITING ENDURANCE BEAM STRENGTH LOAD, F_o, is based on the Lewis equation without a velocity factor.

$$F_o = \frac{s_o b y \pi \cos\psi}{P_d}$$

where the symbols are the same as above.

The limiting endurance strength load, F_o, must be equal to or greater than the dynamic load F_d.

THE LIMITING WEAR LOAD, F_w, for helical gears may be determined by the Buckingham equation for wear.

$$F_w = \frac{D_p b Q K}{\cos^2\psi}$$

where D_p = pitch diameter of pinion, in.

$Q = \dfrac{2D_g}{D_p + D_g} = \dfrac{2N_g}{N_p + N_g}$ (N_p and N_g are actual numbers of teeth)

$K = s_{es}^2(\sin\phi_n)(1/E_p + 1/E_g)/1.4$

s_{es} = surface endurance limit. (See Chapter 18 on spur gears.)

The limiting wear load F_w must be equal to or greater than the dynamic load F_d.

THE DYNAMIC LOAD, F_d, for helical gears is the sum of the transmitted load and an incremental load due to dynamic effects:

$$F_d = F + \frac{0.05V(Cb\cos^2\psi + F)\cos\psi}{0.05V + \sqrt{Cb\cos^2\psi + F}}$$

where the symbols are the same as before. Values for C, which is a function of the amount of effective error in tooth profiles, may be obtained from Chapter 18 on spur gears.

F_w must be $\geq F_d$ and F_o must be $\geq F_d$

Note that F_o and F_w are allowable values which cannot be exceeded.

SOLVED PROBLEMS

1. For a helical gear derive an expression for the virtual number of teeth N_f in terms of the helix angle ψ and the actual number of teeth N.

Solution:

Fig. 19-3 below shows one tooth of a helical gear of pitch diameter D. Consider section A–A in the normal plane. This section will be that of an ellipse whose minor diameter is D. The radius of curvature at point B is

$$r = \frac{D}{2\cos^2\psi} \quad \text{(from analytical geometry)}$$

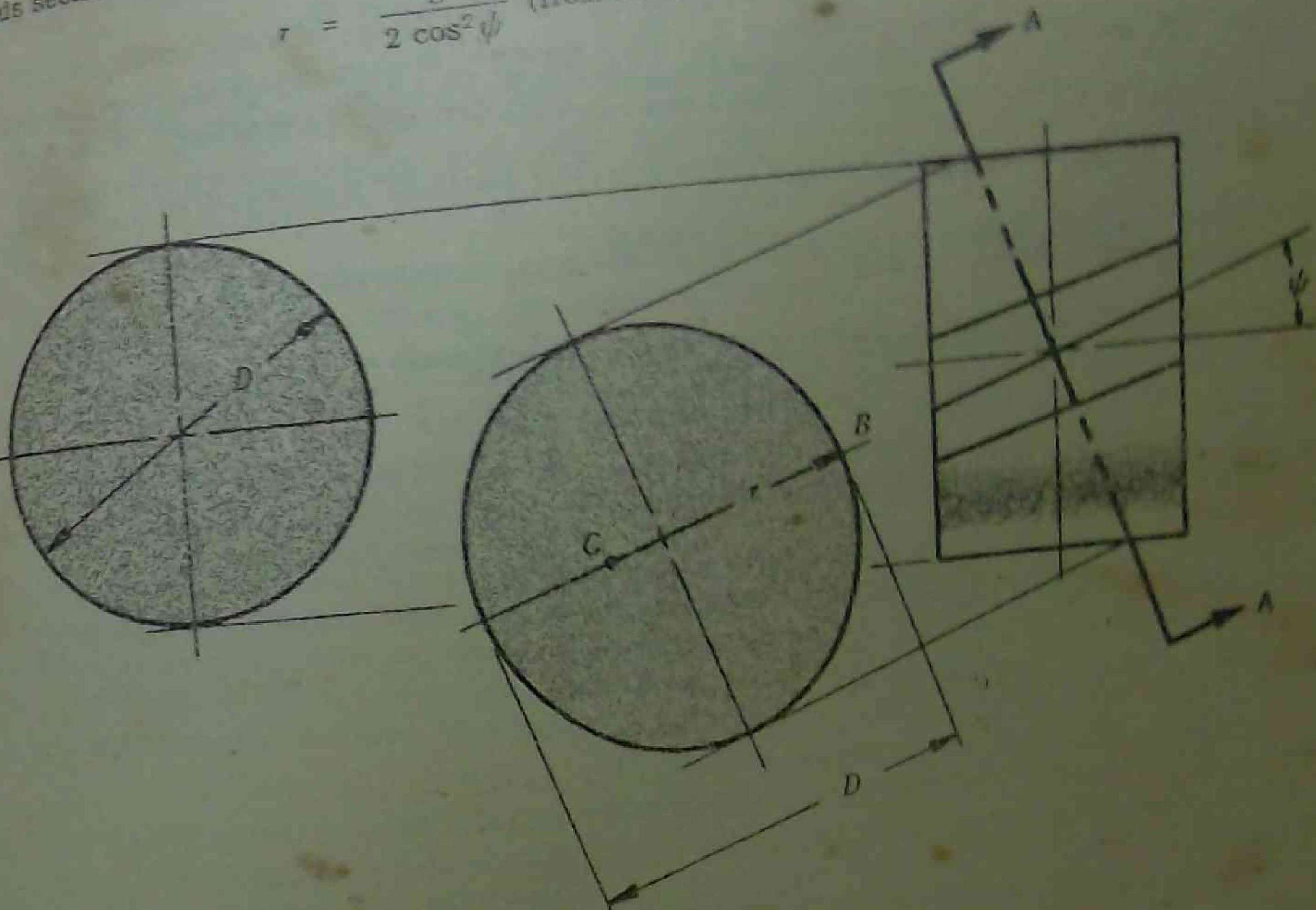


Fig. 19-3

The shape of the tooth at B would be that of one generated on the surface of a pitch cylinder of radius r, and the number of teeth on this surface is defined as the virtual or formative number of teeth N_f.

$$N_f = \frac{2\pi r}{P_{nc}} = 2rP_{nd} = \frac{2rP_d}{\cos\psi} = \frac{2DP_d}{2\cos^3\psi} = \frac{N}{\cos^3\psi}$$

2. A pair of helical gears are to transmit 20 hp. The teeth are 20° stub in diametral plane and have a helix angle of 45°. The pinion has a 3 inch pitch diameter and operates at 10,000 rpm. The gear has a 12 inch pitch diameter. If the gears are made of cast steel (SAE 1235, s_o = 15,000 psi), determine a suitable diametral pitch and face width. The pinion is heat treated to a brinell of 300 and the gear has a brinell hardness of 200.

Solution:

(a) For a strength check, s_o = 15,000 psi. Since the diameters are known, we find

$$\frac{P_d^2}{y} = \frac{s_o k \pi^2 \cos\psi}{F}\left(\frac{78}{78+\sqrt{V}}\right) = 3500 \text{ (allowed)}$$

where V = 7850 fpm, assumed k = 5, $\cos\psi$ = 0.707, and F = 20(33,000)/7850 = 84 lb.

[illegible]ker gear since the two gears are of the same material.

If $y = 0.15$, $P_d = 24$, try $P_d = 24$. Now $N_g = 3(24) = 72$, $N_f = N/\cos^3\psi = 72/(0.707)^3 = 204$, $y = 0.168$ (approximated by using a 20° pressure angle), and $P_d^2/y = (24)^2/0.168 = 3430$ which is satisfactory since the allowed value is 3500.

The k value may be reduced to $k = 6(3430/3500) = 5.9$. Then $b = 5.9(\pi/24) = 0.772$ in.; use $\frac{7}{8}$ in. face width.

(b) For a dynamic check determine the wear load F_w, the endurance beam load F_o, and compare with the dynamic load F_d.

$$F_w = \frac{D_p b Q K}{\cos^2\psi} = 803 \text{ lb (allowed)}$$

where $\tan\phi_n = \tan 20° \cos 45°$, $\phi_n = 14.43°$; $b = 0.875$ in.; $D_p = 3$ in.; $Q = 2D_g/(D_p + D_g) = 2(12)/(3+12) = 1.6$; $s_{es} = 90,000$ psi since the average brinell number of the two gears is 250; and

$$K = s_{es}^2(\sin\phi_n)(1/E_p + 1/E_g)/1.4 = (90,000)^2(\sin 14.43°)[2/(30\times10^6)]/1.4 = 95.5.$$

$$F_o = \frac{s_o b y \pi \cos\psi}{P_d} = 204 \text{ lb (allowed)}$$

$$F_d = F + \frac{0.05V(Cb\cos^2\psi + F)\cos\psi}{0.05V + \sqrt{Cb\cos^2\psi + F}} = 391 \text{ lb}$$

where $s_o = 15,000$ psi, $b = 0.875$ in., $y = 0.168$, $P_d = 24$, $F = 84$ lb, $V = 7850$ fpm, and $C = 800$ for a precision cut gear having a profile error of 0.0005 in.

The preliminary design of $P_d = 24$ is not satisfactory since F_d is greater than F_o. However, the design is satisfactory from the standpoint of wear since the wear load is greater than the dynamic load. If the same materials are retained, it will be necessary to reduce the pitch. Successive trials establish a required standard pitch of $P_d = 14$. Now let $b = 6\pi/14 = 1.345$ in.; use $b = 1.25$ in. Then $N_g = 3(14) = 42$, $N_f = 42/(0.707)^3 = 119$, $y = 0.152$ (approximated by using a 20° pressure angle), and we obtain $F_o = 451$ lb and $F_d = 411$ lb. This a satisfactory solution.

3. A pair of helical gears with a 23° helix angle is to transmit $3\frac{1}{2}$ hp at 10,000 rpm of the pinion. The velocity ratio is 4 to 1. Both gears are to be made of hardened steel, with an allowable $s_o = 15,000$ psi for each gear. The gears are 20° stub and the pinion is to have 24 teeth. Determine the minimum diameter gears that may be used, and the required BHN.

Solution:

(a) Check for strength first. The pinion is the weaker of the two mating gears.

Since the diameters are unknown, use the following form of the Lewis equation and find

$$s = \frac{2TP_d^2}{k y \pi^2 N \cos\psi} = 0.242 P_d^2$$

where the torque on the pinion is $T = 3.5(63\,000)/10\,000 = 22$ in-lb, assumed $k = 6$, $N = 24$, $N_f = N/\cos^3\psi = 31$, $y = 0.120$ (approximated by using a 20° pressure angle), and $\psi = 23°$.

Assuming a velocity factor $= \frac{1}{2}$, $0.242P_d^2 = \frac{1}{2}(15,000)$ and $P_d = 31$. Try a standard pitch of 32; then $D_p = 24/32 = 0.75$ in. and $V = 0.75\pi(10,000)/12 = 1965$ fpm.

$$s_{allowable} = 15,000\left(\frac{78}{78 + \sqrt{1965}}\right) = 9550 \text{ psi}, \qquad s_{induced} = 0.242(32)^2 = 7920 \text{ psi}$$

The design is satisfactory from the standpoint of strength.

The k value may be reduced to $k = 6(7920/9550) = 5$. Now $b = 5\pi/32 = 0.49$; use $b = \frac{1}{2}$ in.

(b) Check for wear load and endurance beam load. Assume average brinell hardness of 250 for first trial.

$$F_w = \frac{D_p b Q K}{\cos^2\psi} = \frac{(0.75)(0.5)(1.6)(128)}{\cos^2 23°} = 96.7 \text{ lb (allowed)}$$

$$F_o = \frac{s_o b y \pi \cos\psi}{P_d} = \frac{(15{,}000)(0.5)(0.139)\pi(0.92)}{32} = 94.9 \text{ lb (allowed)}$$

where $\tan\phi_n = \tan 20^\circ \cos 23^\circ$, $\phi_n = 18.5^\circ$, $Q = 2(96)/(24+96) = 1.6$, $K = 122.0$ (based on ϕ_n).

(c) Check for dynamic load. Assume $C = 860$ (precision cut gears).

$$F_d = F + \frac{0.05V(Cb\cos^2\psi + F)\cos\psi}{0.05V + \sqrt{Cb\cos^2\psi + F}} = 280.6 \text{ lb}$$

where $F = 22/0.375 = 58.6$ lb, $b = 0.5$, $V = 1965$ fpm, $\cos 23^\circ = 0.92$.

The design is unsatisfactory from the standpoint of wear, since the dynamic load is greater than the wear load and the endurance strength beam load. It will probably be necessary to increase the diameters and change the material to one having a higher surface endurance limit.

Starting with the wear load equation, $F_w = D_p b(1.6)K/(0.92)^2$. Let $b = 6P_c = \pi D_p/4$ for a 24 tooth pinion, then $F_w = 1.485 D_p^2 K$.

The dynamic load equation may also be set up in terms of the pinion diameter using $F = 2T/D_p = 44/D_p$, $0.05V = 131 D_p$ for 10,000 rpm, and assuming $C = 860$.

$$F_d = \frac{44}{D_p} + \frac{131D_p[(860\pi D_p/4)(0.92)^2 + 44/D_p](0.92)}{131D_p + \sqrt{(860\pi D_p/4)(0.92)^2 + 44/D_p}}$$

Now by successive trials determine a satisfactory combination of D_p and K. Try $P_d = 10$ or $D_p = 2.4$ in., $K = 250$, $b = \pi(2.4)/4 = 1.885$; then

$$F_w = 1.485 D_p^2 K = 2140 \text{ lb}, \quad F_d = 1164 \text{ lb}, \quad F_o = (15{,}000)(1.885)(0.139)\pi(0.92)/10 = 1128 \text{ lb}$$

This will give a solution, since F_d is only slightly (2.7%) greater than F_o.

4. A pair of precision cut helical steel gears on parallel shafts with a center distance of 7.5 in. transmits power with a velocity ratio of 4 to 1. The pinion rotates at 10,000 rpm. Both gears are made from the same material, having $s_o = 15{,}000$ psi. The teeth are 20° stub, with a 45° helix angle. The face width is 0.75 in. and the diametral pitch is 24. Determine the maximum horsepower that can be transmitted safely, considering wear and strength. Both gears have a brinell hardness of 400.

Solution:

(a) Determine the wear load.

$$F_w = \frac{D_p b Q K}{\cos^2\psi} = \frac{3(0.75)(1.6)(268)}{(0.707)^2} = 1929 \text{ lb}$$

where $D_p = 3$, $b = 0.75$, $Q = 1.6$, $\tan\phi_n = \tan 20^\circ \cos 45^\circ$, $\phi_n = 14.43^\circ$, and

$$K = s_{es}^2 (\sin\phi_n)(1/E_p + 1/E_g)/1.4 = (150{,}000)^2(0.25)[2/(30\times10^6)]/1.4 = 268$$

(b) Determine the endurance load.

$$F_o = \frac{s_o b y \pi \cos\psi}{P_d} = \frac{(15{,}000)(0.75)(0.139)\pi(0.92)}{24} = 188.5 \text{ lb}$$

(c) The dynamic load may not exceed 188.5 lb. $V = 7850$ ft/min.

$$F_d = F + \frac{(0.05)(7850)(860\times0.75\times0.92^2 + F)(0.92)}{(0.05)(7850) + \sqrt{860(0.75)(0.92)^2 + F}}$$

Putting $F = 0$ in this equation, we obtain $F_d = 474$ lb, which is greater than F_o.

Buckingham equations indicate that the above design will not be satisfactory from the standpoint of endurance even with zero horsepower. Based on average load conditions, the gear life would be limited.

Bevel Gears

BEVEL GEARS are usually used to connect intersecting shafts, and may be classified according to the magnitude of their pitch angle. Those having a pitch angle α less than 90° are external bevels as shown in Fig. 20-1 below. Those having a pitch angle of 90° are called crown gears as shown in Fig. 20-2 below. Those having a pitch angle α greater than 90° are internal bevels as shown in Fig. 20-3 below. The sum of the pitch angles of two mating bevel gears is equal to the angle between the intersecting shafts. The Hypoid bevel gear is used to connect non-intersecting shafts. The teeth may be either straight or spiraled with respect to the cone element. The discussion in this chapter will be limited to straight tooth bevel gears connecting shafts intersecting at 90° as shown in Fig. 20-4 below.

Fig. 20-1

Fig. 20-2

Fig. 20-3

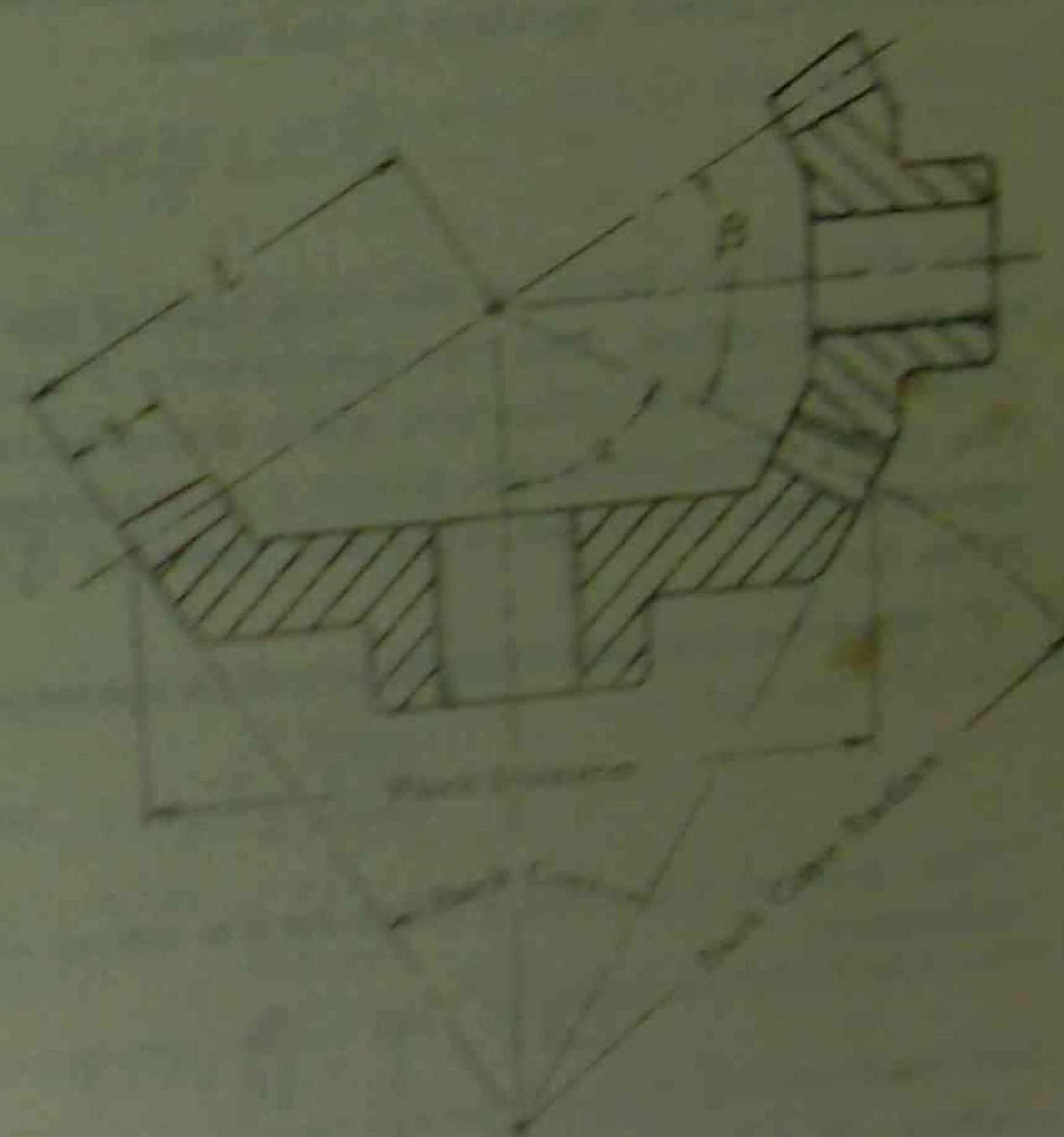


Fig. 20-4

As is true in the design of most other machine elements, there are numerous approaches to gear design. No firm rules can be established since there are so many variables. Most engineers follow procedures by Buckingham, Gleason, or those recommended by the AGMA (American Gear Manufacturers Association). Any procedure for gear design should be considered as preliminary until proved by experiment to satisfy specific requirements.

As for the case of spur and helical gears, design of bevel gears is based on beam strength, dynamic load, and wear.

STRENGTH DESIGN of a straight tooth bevel gear may be based on the Lewis equation. It should be noted that the tooth tapers and becomes smaller in cross section as it converges to the apex of the cone. The Lewis equation is modified as follows to correct for this situation.

The permissible force F that may be transmitted is

$$F = \frac{sby\pi}{P_d}\left(\frac{L-b}{L}\right)$$

where s = allowable bending stress, psi

y = form factor based on the formative number of teeth and the type of tooth profile

L = the cone distance, in., and is equal to the square root of the sum of the squares of the pitch radii of the mating gears (for shafts intersecting at 90°)

b = the face width of the gear, in.

P_d = the diametral pitch based on the largest tooth cross section.

For ease of manufacture and satisfactory operation of bevel gears, it is recommended that the face width be limited to between $L/3$ and $L/4$, where L is the cone distance. In general we will design the face width close to, but never greater than, $L/3$. When designing for strength the diameter of the gear may be either known or unknown. When the diameter is known, it is convenient to use the modified Lewis equation in this form:

$$\frac{P_d}{y} = \frac{sb\pi}{F}\left(\frac{L-b}{L}\right) = \text{an allowed value}$$

Note that all the terms on the right side of the above equation can be determined after the material has been specified. The transmitted force F is determined by dividing the torque on the weaker gear by its pitch radius. The face width b may be taken as $L/3$ rounded out to the nearest 1/8 in. under this value. The allowable stress s is evaluated as explained in the following section. The above equation then yields an allowable value of P_d/y which must be satisfied by selecting a suitable value for P_d.

When the diameter is unknown, it is convenient to use the following form of the Lewis equation:

$$s = \frac{2TP_d^2}{b\pi yN}\left(\frac{L}{L-b}\right) = \text{actual stress} \leq \text{allowed stress}$$

This equation will yield a value for the actual stress in terms of P_d^3 after making the following substitutions:

$$\text{Let } b = \frac{L}{3} = \frac{N_p}{6P_d}\sqrt{1+R^2} \qquad \text{and} \qquad \text{Let } \frac{L}{L-b} = \frac{3}{2}$$

where N = actual number of teeth on the weaker gear

N_p = number of teeth on the pinion

R = ratio of the angular velocity of the pinion to the angular velocity of the gear.

Design from a standpoint of strength may be considered as a first approximation which must be checked for wear and dynamic effects as explained later.

THE ALLOWABLE STRESSES, s, for the average conditions may be taken as

$$s = s_o\left(\frac{1200}{1200+V}\right) \text{ psi for cut teeth} \qquad \text{or} \qquad s = s_o\left(\frac{78}{78+\sqrt{V}}\right) \text{ psi for generated teeth}$$

where s_o is the endurance limit of the gear material for released loading, corrected for average stress concentration, psi. An approximate value for s_o is 1/3 of the ultimate strength, based on an average value for stress concentration. V is the pitch line velocity, fpm.

THE FORMATIVE OR VIRTUAL number of teeth, N_f, for a bevel gear is the number of teeth, having the same pitch as the actual gear, that could be cut on a gear having a pitch radius equal to the radius of the back cone.

$$N_f = N/\cos\alpha$$

where N = actual number of teeth on the gear and α = pitch angle or half of the cone angle.

THE LIMITING WEAR LOAD, F_w, may be approximated from

$$F_w = \frac{0.75\,D_p bKQ}{\cos\alpha} \quad \text{(an allowed value)}$$

where D_p, b, K, and Q are the same as for spur gears, except that Q is based on the formative number of teeth, and α is the pitch angle of the pinion.

THE LIMITING ENDURANCE LOAD, F_o, may be approximated from

$$F_o = \frac{s_o b y \pi}{P_d}\left(\frac{L-b}{L}\right) \quad \text{(an allowed value)}$$

THE DYNAMIC LOAD, F_d, which is the transmitted load plus an incremental load due to dynamic effects, may be approximated from

$$F_d = F + \frac{0.05V(bC+F)}{0.05V + \sqrt{bC+F}}$$

where the symbols are the same as for spur gears. F_d must be $\leq F_w$. F_d must be $\leq F_o$.

Note that F_o and F_w are allowed values which must not be exceeded by the dynamic load.

THE AMERICAN GEAR MANUFACTURERS ASSOCIATION (AGMA) STANDARDS recommend the following horsepower rating, for peak load, for both straight and spiral bevel gears:

$$hp = \frac{snD_p b y \pi(L-0.5b)}{126{,}000 P_d L}\left(\frac{78}{78+\sqrt{V}}\right)$$

where s = 250 times the Brinell hardness number of the weaker gear for gears hardened and also not hardened after cutting.

s = 300 times the Brinell hardness number of the weaker gear if the gear is case hardened.

n = speed of pinion, rpm.

All other symbols are the same as before.

THE AGMA STANDARDS FOR WEAR (DURABILITY) recommend the following horsepower ratings:

$$hp = 0.8\,C_m C_B b \quad \text{for straight bevel gears}$$

$$hp = C_m C_B b \quad \text{for spiral bevel gears}$$

where C_m = a material factor as listed below

$$C_B = \frac{D_p^{1.5}\,n}{233}\left(\frac{78}{78+\sqrt{V}}\right)$$

n = rpm of the pinion.

2. A spring clip, made from a 1" diameter rod, is shown in Fig. 4-3. Determine the maximum shear stress and specify its location or locations.

Solution:

(a) The location of the maximum normal stress can be found from inspection and comparison. Any free body of the bar which includes the two applied forces of the upper or lower parts would have the same bending moment since the two applied forces give a pure couple. The maximum bending stress will occur at the inside fiber of the sections with the inside radius of curvature $r_i = 3''$ (from A to B and C to D). The maximum stress will not occur where $r_i = 4''$.

(b) $r_n = \dfrac{[r_o^{1/2} + r_i^{1/2}]^2}{4} = \dfrac{[4^{1/2} + 3^{1/2}]^2}{4} = 3.482''$

(c) $e = R - r_n = 3.5 - 3.482 = .018''$
$h_i = .50 - .018 = .482$

(d) $s_i = \dfrac{Mh_i}{Aer_i} = \dfrac{500(.482)}{\frac{1}{4}\pi(1)^2(.018)(3)} = 5680$ psi

(e) The stress from A to B is tension and from C to D is compression.

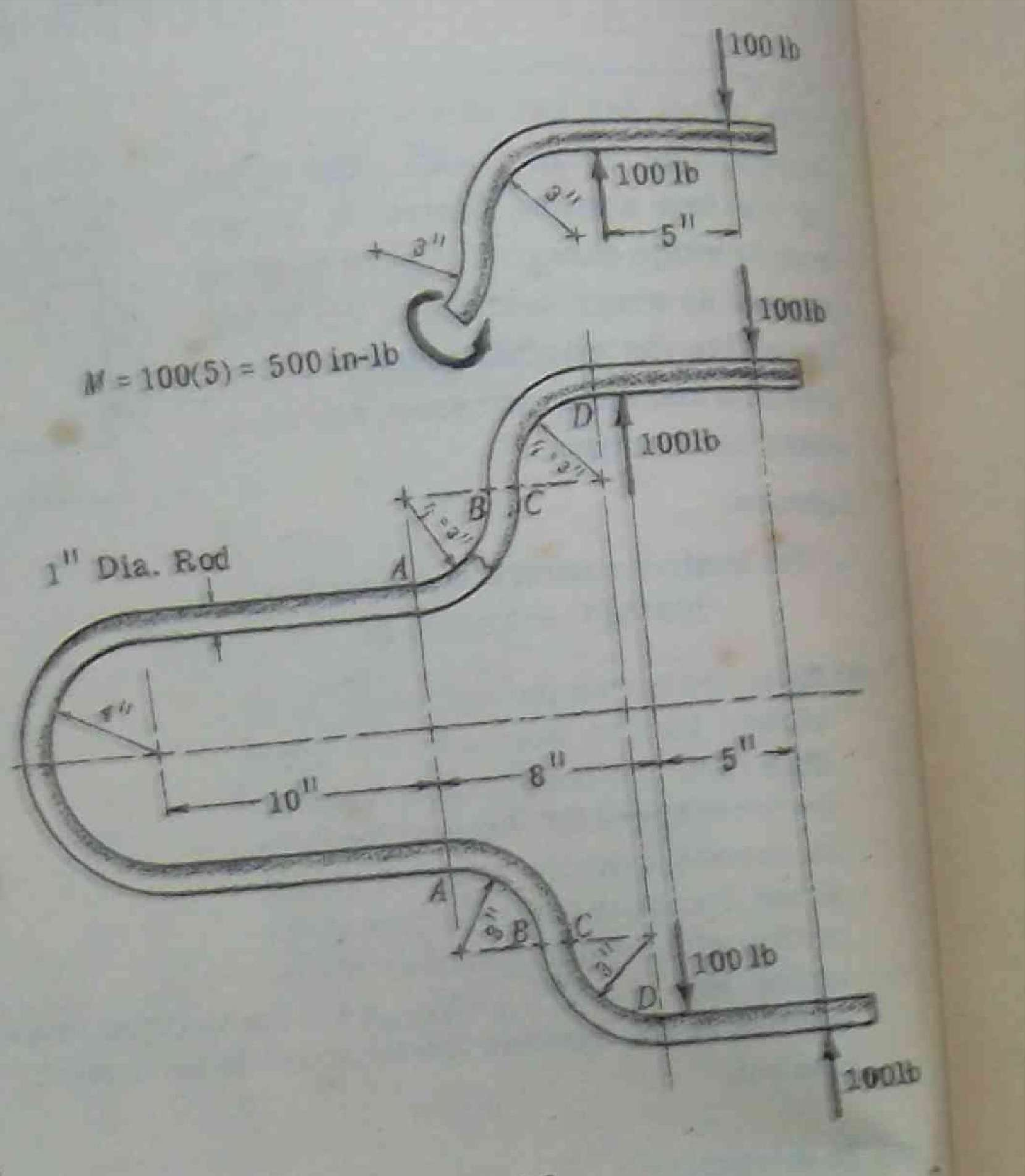


Fig. 4-3

(f) The maximum shear stress is $\frac{1}{2}$ (5680) = 2840 psi and occurs at every point from A to B and from C to D.

(g) Since the section is symmetrical, the maximum stress at the outer fibers need not be checked.

3. An open S link is made from a 1" diameter rod. Determine the maximum tensile stress and maximum shear stress.

Solution:

(a) A comparison of section A-A and B-B, in Fig. 4-4, shows that the bending moment at A-A is less than at section B-B, but the radius of curvature is less at A-A than B-B. It will be necessary to investigate both sections.

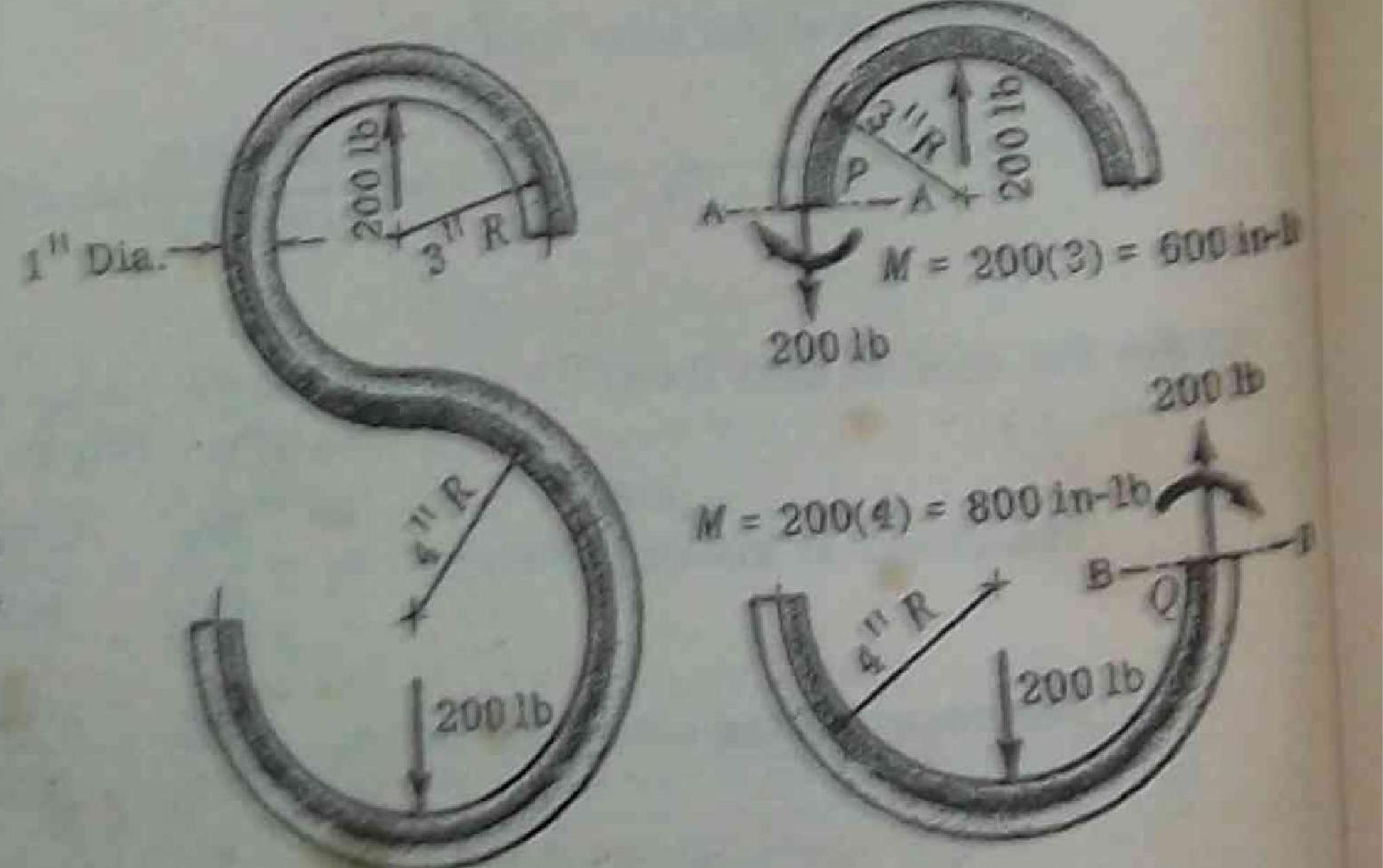


Fig. 4-4

__At Section A-A, point P:__

(b) $M = 600$ in-lb

(c) $r_n = \dfrac{[r_o^{1/2} + r_i^{1/2}]^2}{4} = \dfrac{[(3\frac{1}{2})^{1/2} + (2\frac{1}{2})^{1/2}]^2}{4} = 2.979''$

$e = R - r_n = 3.0 - 2.979 = .021''$, $\qquad h_i = .5 - .021 = .479''$

Bending stress + direct tension $= \dfrac{Mh_i}{Aer_i} + \dfrac{P}{A} = \dfrac{600(.479)}{\frac{1}{4}\pi(1)^2(.021)(2.5)} + \dfrac{200}{\frac{1}{4}\pi(1)^2} = 6960 + 260 = 7220$ psi tension

PARTIAL LIST, MATERIAL FACTOR C_m

Gear		Pinion		
Material	Brinell	Material	Brinell	C_m
I	160-200	II	210-245	0.30
II	245-280	II	285-325	0.40
II	285-325	II	335-360	0.50
II	210-245	III	500	0.40
II	285-325	IV	550	0.60
III	500	IV	550	0.90
IV	550	IV	550	1.00

I = Annealed steel   III = Oil or surface-hardened steel
II = Heat-treated steel   IV = Case-hardened steel

It has been found by experience that generally, if cast iron teeth are strong enough they will not fail by wear, and that if steel teeth satisfy the wear requirements they will be strong enough.

SOLVED PROBLEMS

1. A cast iron bevel gear has a pitch diameter D of 24 in. and a pitch angle α of 30°. The diametral pitch P_d is 10. Determine the permissible endurance load F_o. The teeth are 20° full depth.

Solution:

$$F_o = \frac{s_o b y \pi}{P_d}\left(\frac{L-b}{L}\right) = \frac{(8000)(8)(0.149)\pi}{10}\left(\frac{24-8}{24}\right) = 1996 \text{ lb}$$

where $s_o = 8000$ psi for cast iron, $L = D/(2 \sin 30^\circ) = 24$ in., $b = L/3 = 8$ in.,

$y = 0.149$ (form factor based on $N_f = \dfrac{N}{\cos\alpha} = \dfrac{P_d D}{\cos\alpha} = \dfrac{10(24)}{\cos 30^\circ} = 277$ teeth).

2. Two steel bevel gears, both having a brinell hardness of 250, connect shafts at 90°. The teeth are $14\frac{1}{2}^\circ$ full depth and the diametral pitch is 6. The number of teeth on the pinion and gear are 30 and 48. The face width is 1.5 in. Determine the wear load F_w.

Solution:

$$F_w = \frac{0.75 D_p b K Q}{\cos\alpha} = \frac{(0.75)(5)(1.5)(96)(1.44)}{0.846} = 918 \text{ lb}$$

where $D_p = N_p/P_d = 30/6 = 5$ in., $D_g = N_g/P_d = 48/6 = 8$ in., $b = 1.5$ in

$$K = \frac{s_{es}^2(\sin\delta)(1/E_p + 1/E_g)}{1.4} = \frac{(90{,}000)^2(0.25)[2/(30\times 10^6)]}{1.4} = 96$$

$s_{es} = (\text{BHN})(250) - 10{,}000 = (400)(250) - 10{,}000 = 90{,}000$ psi

$$Q = \frac{2N_f\,(\text{gear})}{N_f\,(\text{pinion}) + N_f\,(\text{gear})} = \frac{2(90.5)}{35.4 + 90.5} = 1.44$$

$$N_f\,(\text{gear}) = \frac{N_g}{\cos\alpha\,(\text{gear})} = \frac{48}{0.53} = 90.5, \quad N_f\,(\text{pinion}) = \frac{N_p}{\cos\alpha\,(\text{pinion})} = \frac{30}{0.847} = 35.4$$

$$\cos\alpha\,(\text{gear}) = \frac{R_p}{L} = \frac{2.5}{4.72} = 0.53, \quad \cos\alpha\,(\text{pinion}) = \frac{R_g}{L} = \frac{4}{4.72} = 0.847, \quad L = \sqrt{R_p^2 + R_g^2} = 4.72$$

3. Two cast iron bevel gears transmit 3 hp with a pitch line velocity of 860 fpm. The face width of the gears is 0.75 in. Determine the dynamic load F_d. The teeth are $14\frac{1}{2}°$ involute and precision cut.

Solution:

$$F_d = F + \frac{0.05V(Cb+F)}{0.05V + \sqrt{Cb+F}} = 115 + \frac{(0.05)(860)(400 \times 0.75 + 115)}{(0.05)(860) + \sqrt{400 \times 0.75 + 115}} = 397 \text{ lb}$$

where $F = (hp)(33{,}000)/V = (3)(33{,}000)/860 = 115$ lb

$C = 400$ for precision cut gears (same as for spur gears)

$V = 860$ fpm, $b = 0.75$ in.

4. Two cast iron bevel gears having pitch diameters of 3 in. and 4 in. respectively are to transmit 3 hp at 1100 rpm of the pinion. The tooth profiles are of $14\frac{1}{2}°$ composite form.

(a) Determine the face width b and the required P_d from the standpoint of strength using the Lewis equation.

(b) Check design from the standpoint of dynamic load and wear.

Solution:

The pinion is the weaker since both gear and pinion are of the same material.

(a) Design for strength.

$$\frac{P_d}{y} = \frac{sb\pi}{F}\left(\frac{L-b}{L}\right) = \frac{(4650)(0.75)\pi}{115}\left(\frac{2.5-0.75}{2.5}\right) = 66.5 \quad \text{(allowed)}$$

where $s = s_o\left(\frac{1200}{1200+V}\right) = 8000\left(\frac{1200}{1200+860}\right) = 4650$ psi. $V = \frac{3\pi}{12}(1100) = 860$ fpm

$L = \sqrt{R_p^2 + R_g^2} = \sqrt{1.5^2 + 2^2} = 2.5$ in.

$b = L/3 = 2.5/3 = 0.83$ in. (use 0.75 in.).

$s_o = 8000$ psi for cast iron. $F = (3)(33{,}000)/860 = 115$ lb

Then if $y = 0.1$ and $P_d/y = 66.5$, $P_d = 6$ or 7.

Try $P_d = 7$: $N_p = (3)(7) = 21$, N_f (pinion) $= 21/\cos\alpha = 21/(2/2.5) = 26.2$ (formative teeth). $y = 0.098$ for 26.2 teeth. Now $P_d/y = 7/0.098 = 71.5$ (too weak since $71.5 > 66.5$).

Next try $P_d = 6$: $N_p = (3)(6) = 18$, N_f (pinion) $= 18/\cos\alpha = 18/(2/2.5) = 22.5$. $N_g = (4)(6) = 24$. N_f (gear) $= 24/\cos\alpha = 24/(1.5/2.5) = 40$, $y = 0.094$ for 22.5 teeth. Now $P_d/y = 6/0.094 = 63.8$ (satisfactory since $63.8 < 66.5$).

(b) Check for wear and dynamic effects using $P_d = 6$.

$$F_w = \frac{0.75 D_p bKQ}{\cos\alpha \text{ (pinion)}} = \frac{(0.75)(3)(0.75)(193)(1.28)}{0.8} = 522 \text{ lb} \quad \text{(allowed)}$$

where $K = 193$ based on $s_{es} = 90{,}000$ psi for cast iron on cast iron (Table III, Chap. 18), and

$$Q = \frac{2N_f \text{ (gear)}}{N_f \text{ (pinion)} + N_f \text{ (gear)}} = \frac{(2)(40)}{22.5+40} = 1.28.$$

$$F_o = \frac{s_o by\pi}{P_d}\left(\frac{L-b}{L}\right) = \frac{(8000)(0.75)(0.094)\pi}{6}\left(\frac{2.5-0.75}{2.5}\right) = 207 \text{ lb} \quad \text{(allowed)}$$

Both F_o and F_w must be equal to or greater than F_d.

$$F_d = F + \frac{0.05V(Cb+F)}{0.05V + \sqrt{Cb+F}} = 115 + \frac{(0.05)(860)(400 \times 0.75 + 115)}{(0.05)(860) + \sqrt{400 \times 0.75 + 115}} = 397 \text{ lb}$$

The design is satisfactory from the standpoint of wear, but the dynamic load is greater than the endurance load. Better material should be specified for the pinion in order to increase the endurance load to at least 397

Chapter 22

Rolling Bearings

INTRODUCTION. Rolling bearing application involves the proper selection, mounting, lubrication, and possibly shielding in order that the bearings function satisfactorily under specified operating conditions.

The selection of a rolling bearing is made from a manufacturer's catalog. Unfortunately, the catalogs of different manufacturers do not necessarily use the same methods of arriving at a bearing selection, principally because of differences in interpretation of test data and service conditions. However, the rating of bearings is based upon certain general theory as outlined in this chapter and modified by various companies according to their experiences.

The mounting of bearings may be based on one of several recommended procedures, the arrangement used quite often being controlled by the economics.

Rolling bearings are also called anti-friction bearings, although the friction in rolling bearings is comparable to that in well designed journal bearings operating under thick film conditions. The decision as to the kind of bearing to use, that is, whether to use a rolling bearing or journal bearing, can be influenced by one or several of the following:

(1) Rolling bearings have an advantage where starting torques are high because of the rolling action of the balls, or rollers.

(2) Rolling bearings, especially at high speeds, are not as quiet in operation as journal bearings.

(3) Where space limitations are present, rolling bearings are preferable if the axial dimension is limited; and journal bearings are preferable if the radial dimension is limited, although the use of a ring or collar oiled bearing with the oil reservoir might require a large radial dimension.

(4) Where electrical insulation is desirable, the oil film in perfect lubrication will help provide insulation.

(5) Rolling bearings give warning (by becoming noisy) when failure is imminent; whereas, when failure occurs in journal bearings, the failure is sudden with more disastrous results.

(6) Rolling bearings can take a combination of radial and thrust loads (except for straight roller bearings).

(7) Rolling bearings can be preloaded, when desirable, to reduce deflections in the bearing and to provide for more accuracy, as in machine tools.

(8) Clearances in rolling bearings need be much less than in journal bearings, providing for accurate positioning of machine parts such as gears.

(9) Rolling bearings can be prepacked with grease to provide for a maintenance-free installation. Where oil is used for lubrication in rolling bearings, the lubrication problem is usually much simpler than for journal bearings. Failure of the lubricating system with rolling bearings is not calamitous, as it might be with journal bearings.

(10) Rolling bearings can take high overloads for short periods.

COEFFICIENT OF FRICTION in rolling bearings varies with speed, load, amount of lubrication, assembly, temperature of operation. A constant coefficient can be used for approximate calculations under favorable lubrication and what might be called normal operating conditions. The values listed are as recommended by SKF Industries, Inc.:

f = 0.0010 for self aligning bearings (radial load)
f = 0.0011 for cylindrical roller bearings with flange-guided short rollers (radial load)
f = 0.0013 for thrust ball bearings (thrust load)
f = 0.0015 for single row ball bearings (radial load)
f = 0.0018 for spherical roller bearings (radial load)
f = 0.0018 for tapered roller bearings

The coefficients of friction due to use of oils of high viscosity, more than the optimum amount of lubrication, or new bearings will be greater than those listed. Seal frictions should not be ignored. The values of coefficient of friction, as found by tests by New Departure, have been found to vary from 0.0005 to 0.003, with a general average of about 0.001.

It should be pointed out that improper assembly, as might occur with an interference between the shaft and inner race greater than recommended by the bearing manufacturers, can cause excessive binding and excessive friction.

The friction torque is given by

$$M_t = Ff(D/2)$$

where M_t = friction torque, in-lb; F = radial or axial load as specified, lb; f = coefficient of friction; D = diameter of the *bore* of the bearing. (It is usual practice to refer the frictional force to the bore of the bearing, or shaft diameter.)

STATIC CAPACITY OF BEARINGS depends on the conditions subsequent to static loading, as well as the various physical dimensions. The static capacity of a bearing which is *not* rotated subsequently to static loading will be much higher than one which is rotated; very small loads will cause permanent deformations in the rolling element and raceways which may prevent quiet operation at high speed even though friction is not affected appreciably and the bearing is not damaged.

A light, medium, and heavy series bearing of the same bore is shown in Fig. 22-1.

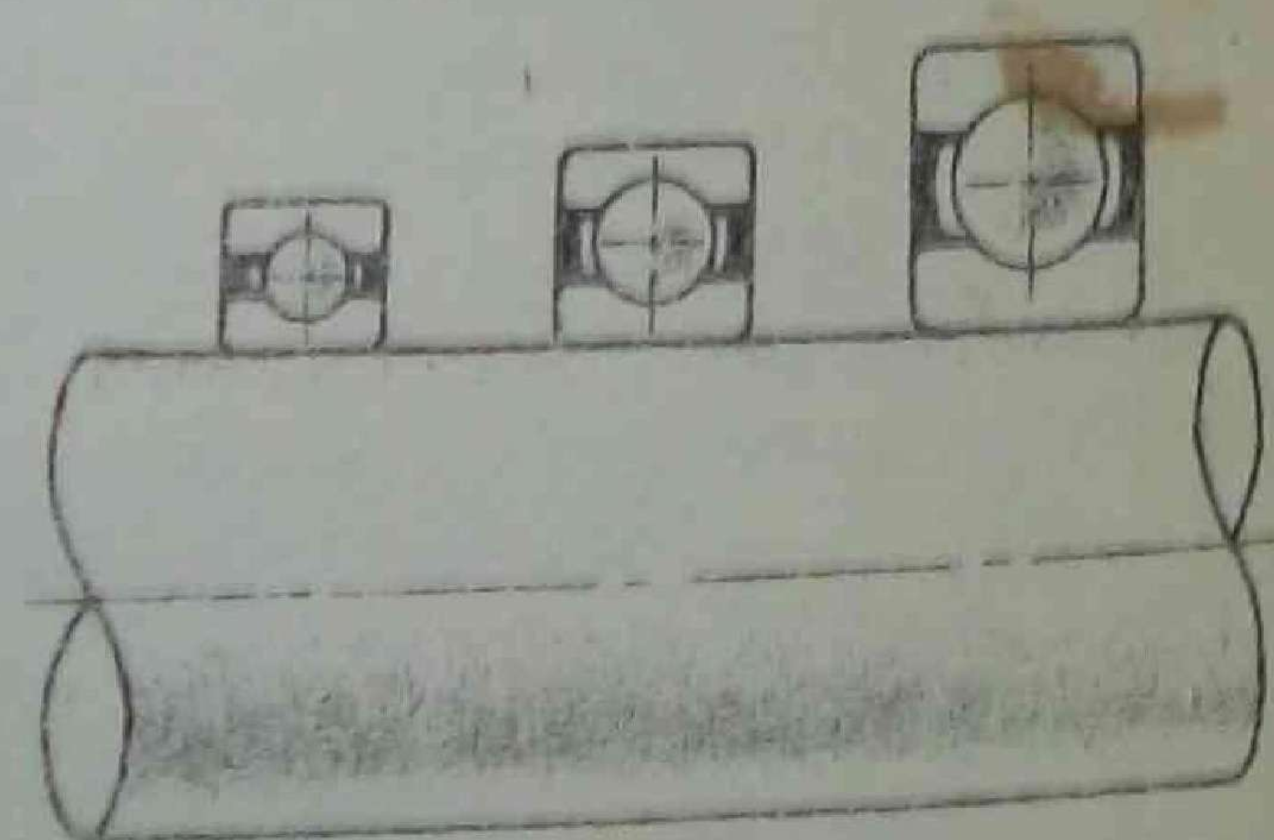


Fig. 22-1

The initial work done by Stribeck on static capacity of bearings served for many years as the basis of rating bearings. Later experience and test data gave added information as to the proper rating of bearings, and modifications were made by Palmgren with subsequent modifications made by the Anti-Friction Bearing Manufacturers Association (AFBMA) to suit dynamic conditions. Stribeck's work still serves as a basis for static rating of bearings.

Stribeck's equation for the static capacity C_o for ball bearings is

$$C_o = \frac{KZD^2}{5}$$

where K is a constant depending upon ball diameter, Z = number of balls, D = diameter of the balls; and for the static capacity of a straight roller bearing the equation is

$$C_o = \frac{KZDL}{5}$$

where K = a constant, Z = numbers of rollers, D = diameter of rollers, L = length of rollers.

The following definitions and data for method of evaluating static load ratings of radial ball bearings is taken from the AFBMA standards - *Methods of Evaluating Load Ratings of Ball Bearings* published ... with permission.

I. METHOD OF EVALUATING STATIC LOAD RATINGS FOR RADIAL BALL BEARINGS

A. Definitions

(1) The *static load* is defined as a load acting on a non-rotating bearing.

(2) Permanent deformations appear in balls and raceways under static load of moderate magnitude and increase gradually with increasing load. The permissible static load is therefore dependent upon the permissible magnitude of permanent deformation.

Experience shows that a total permanent deformation of 0.0001 of the ball diameter, occurring at the most heavily loaded ball and race contact, can be tolerated in most bearing applications without impairment of bearing operation.

In certain applications where subsequent rotation of the bearing is slow and where smoothness and friction requirements are not too exacting, a much greater total permanent deformation can be permitted. Likewise, where extreme smoothness is required or friction requirements are critical, less total permanent deformation may be tolerated.

For purposes of establishing comparative ratings, the *basic static load rating* therefore is defined as that static radial load which corresponds to a total permanent deformation of ball and race at the most heavily stressed contact of 0.0001 of the ball diameter.

In single row angular contact ball bearings the basic static load rating relates to the radial component of that load, which causes a purely radial displacement of the bearing rings in relation to each other.

(3) The *static equivalent load* is defined as that static, radial load which, if applied, would cause the same total permanent deformation at the most heavily stressed ball and race contact as that which occurs under the actual condition of loading.

B. Calculation of Basic Static Load Rating and Static Equivalent Load

(1) **Basic Static Load Rating**: The magnitude of the basic static load rating C_0 is

$$C_0 = f_0 i Z D^2 \cos\alpha$$

where i = number of rows of balls in any one bearing

α = nominal angle of contact — the nominal angle between the line of action of the ball load and a plane perpendicular to the bearing axis

Z = number of balls per row

D = ball diameter

Values of the factor f_0 for different kinds of bearings as commonly designed and manufactured and made of hardened steel are given in Table I-1.

Table I-1

Factor f_0

Bearing Type	f_0 Units kg, mm	f_0 Units pound, inch
Self-aligning ball bearings	0.34	484
Radial and angular contact groove ball bearings	1.25	1780

(2) **Static Equivalent Load**: The magnitude of the static equivalent load P_0, for radial bearings under combined radial and thrust loads, is the greater of

$$P_0 = X_0 F_r + Y_0 F_a$$

$$P_0 = F_r$$

where X_0 = a radial factor F_r = the radial load

Y_0 = a thrust factor F_a = the thrust load

Values of X_0 and Y_0 are given in Table I-2.

Table I-2
Factors X_0 and Y_0

Bearing Type		Single Row Bearings(1)		Double Row Bearings(3)	
		X_0	Y_0	X_0	Y_0
Radial Contact Groove Ball Bearings (1)		0.6	0.5	0.6	0.5
Angular Contact Groove Ball Bearings (2)	$\alpha = 20^\circ$	0.5	0.42	1	0.84
	$\alpha = 25^\circ$	0.5	0.38	1	0.76
	$\alpha = 30^\circ$	0.5	0.33	1	0.66
	$\alpha = 35^\circ$	0.5	0.29	1	0.58
	$\alpha = 40^\circ$	0.5	0.26	1	0.52
Self-aligning Ball Bearings		0.5	$0.22 \cot \alpha$	1	$0.44 \cot \alpha$

Notes: (1) P_0 is always $\geq F_r$

(2) For two similar single row angular contact ball bearings mounted "face-to-face" or "back-to-back", use the values of X_0 and Y_0 which apply to a double row angular contact ball bearing. For two or more similar single row angular contact ball bearings mounted "in tandem", use the values of X_0 and Y_0 which apply to a single row angular contact ball bearing.

(3) Double row bearings are presumed to be symmetrical.

(4) Permissible maximum value of F_a/C_0 depends on the bearing design (groove depth and internal clearance).

DYNAMIC CAPACITY OF A BEARING is based on the fatigue life of the material, contrasted with the static capacity which is based on permanent deformation or brinelling. *It is significant to note that, in general, a bearing rotating at low speed has a higher rating than the static rating since the brinelling that takes place is more evenly distributed; consequently, a greater amount of permanent deformation may be tolerated with rotation.*

Life of a bearing can be defined either in terms of hours of rotation at a certain speed, or life can be defined in terms of number of revolutions. It is necessary to define life in terms of the performance of a group of bearings, since the life of a single bearing cannot be predicted. Bearings are rated on either of two bases, depending upon the manufacturer:

(1) the average life of a group of bearings
(2) the life which 90% of the bearings will reach or exceed. The ratings as given by the AFBMA are based upon a life which 90% of the bearings in a group will reach or exceed.

The longest life of a single bearing is seldom longer than 4 times the average life. The life which 50% of a group of bearings will complete or exceed is approximately 5 times the life which 90% of the bearings will complete or exceed. The maximum life of a single bearing is about 30 to 50 times the minimum life. Thus, where dependability and reliability are essential for a single bearing, larger factors of safety must be used, since there is no way of predicting beforehand how far away from average a given bearing may be.

The specific dynamic capacity C of a bearing is defined as the constant radial load in a radial bearing (or constant thrust load in a thrust bearing) that can be carried for a minimum life of 1,000,000 revolutions (which is equivalent to 500 hours of operation at 33.3 rpm); the minimum life in the definition is that life which 90% of the bearings of a group will reach or exceed. Specific dynamic capacity is based upon the inner ring rotating and the outer ring stationary. (Note that the average life would then correspond to about 5 times as much, or 5,000,000 revolutions, which would correspond to 2500 hours at 33.3 rpm.)

The following information on the method of evaluating dynamic load ratings of radial ball bearings [illegible] formulated by the Anti-Friction Bearing Manufacturers Association, Inc. [illegible] gives the definitions, methods of calculations of the basic load ratings, rating life, and equivalent load, and necessary constants for radial ball bearings, which have been selected to serve as illustration of the procedures used in rating all bearings. Similar information for thrust bearings has been prepared by the AFBMA, but is not given here. [illegible] determine the basic load rating for a given bearing, but the selection of a bearing [illegible] of the results for a series of bearings when the ball diameter and number of balls [illegible]

I. METHOD OF EVALUATING DYNAMIC LOAD RATINGS FOR RADIAL BALL BEARINGS

A. Definitions

(1) The life of an individual ball bearing is defined as the number of revolutions (or hours at some given constant speed) which the bearing runs before the first evidence of fatigue develops in the material of either ring or of any of the rolling elements.

(2) The rating life of a group of apparently identical ball bearings is defined as the number of revolutions (or hours at some given constant speed) that 90 percent of a group of bearings will complete or exceed before the first evidence of fatigue develops. [illegible] life which 50 per cent of the group of ball bearings will complete or exceed is approximately five times the rating life.

(3) The basic load rating is that constant stationary radial load which a group of apparently identical ball bearings with stationary outer ring can endure for a rating life of one million revolutions of the inner ring. In single row angular contact ball bearings, the basic load rating relates to the radial component of the load, which results in a purely radial displacement of the bearing rings in relation to each other.

(4) Load ratings, if given for specific speeds, are to be based on a rating life of [illegible] hours.

(5) The *equivalent load* is defined as that constant stationary radial load which, if applied to a bearing with rotating inner ring and stationary outer ring, would give the same life as that which the bearing will attain under the actual conditions of load and rotation.

B. Calculation of Basic Load Rating, Rating Life and Equivalent Load

(1) It is recognized that revisions of this recommendation may be required from time to time as the result of improvements or new developments.

(2) **Basic Load Rating**: The magnitude of the basic load rating C, for radial and angular contact ball bearings, except filling slot bearings, with balls not larger than 25.4 mm or 1 inch in diameter, is

$$C = f_c (i \cos \alpha)^{0.7} Z^{2/3} D^{1.8};$$

with balls larger than 25.4 mm in diameter when kg and mm units are used,

$$C = f_c (i \cos \alpha)^{0.7} Z^{2/3} \, 3.647 D^{1.4}$$

with balls larger than 1 inch in diameter when pound and inch units are used,

$$C = f_c (i \cos \alpha)^{0.7} Z^{2/3} D^{1.4}$$

where i = number of rows of balls in any one bearing

α = nominal angle of contact = nominal angle between the line of action of the ball load and a plane perpendicular to the bearing axis

Z = number of balls per row

D = ball diameter

f_c = a factor which depends on the units used, the geometry of the bearing components, the accuracy to which the various bearing parts are made and the material.

Values of f_c are obtained by multiplying the value of f_c/f from the appropriate column of Table II-1 by a factor f, covered in Appendix 1 (page 262).

Table II-1

Factor $\frac{f_c}{f}$

$\frac{D \cos \alpha}{d_m}$ (1)	$\frac{f_c}{f}$		
	Single row radial contact, single and double row angular contact groove ball bearings (2)	Double row radial contact groove ball bearings	Self-aligning ball bearings
0.05	0.476	0.451	0.176
0.06	0.500	0.474	0.190
0.07	0.521	0.494	0.203
0.08	0.539	0.511	0.215
0.09	0.554	0.524	0.227
0.10	0.566	0.537	0.238
0.12	0.586	0.555	0.261
0.14	0.600	0.568	0.282
0.16	0.608	0.576	0.303
0.18	0.611	0.579	0.323
0.20	0.611	0.579	0.342
0.22	0.608	0.576	0.359
0.24	0.601	0.570	0.375
0.26	0.593	0.562	0.390
0.28	0.583	0.5[illegible]2	0.402
0.30	0.571	0.541	0.411
0.32	0.558	0.530	0.418
0.34	0.543	0.515	0.420
0.36	0.527	0.500	0.421
0.38	0.510	0.484	0.418
0.40	0.492	0.467	0.412

Notes: (1) d_m denotes the pitch diameter of the ball set. For values of $\frac{D \cos \alpha}{d_m}$ other than given in the table, f_c/f is obtained by linear interpolation.

(2) a. When calculating the basic load rating for a unit consisting of two similar single row radial contact ball bearings in a duplex mounting, the pair is considered as one double row radial contact ball bearing.

b. When calculating the basic load rating for a unit consisting of two similar single row angular contact ball bearings in a duplex mounting, "face-to-face" or "back-to-back", the pair is considered as one double row angular contact ball bearing.

c. When calculating the basic load rating for a unit consisting of two or more similar single row angular contact ball bearings mounted "in tandem", properly manufactured and mounted for equal load distribution, the rating of the combination is the number of bearings to the 0.7 power times the rating of a single row ball bearing. If for some technical reason the unit may be treated as a number of individually interchangeable single row bearings, this footnote (2)c does not apply.

(3) **Rating Life:** The approximate magnitude of the rating life L for ball bearings, except filling slot bearings, is

$$L = (C/P)^3 \text{ million revolutions}$$

where P = the equivalent load.

(4) **Equivalent Load:** The magnitude of the equivalent load P for radial and angular contact ball bearings of conventional types, except filling slot bearings, under combined constant radial and constant thrust loads, is

$$P = XVF_r + YF_a$$

where X = a radial factor, V = a rotation factor, Y = a thrust factor, F_r = the radial load, F_a = the thrust load

Values of X, V and Y are given in Table II-2. The factor V, due to lack of sufficient experimental evidence, is used as a matter of precaution.

Table II-2
Factors X, V and Y

Bearing Type			In Relation to the Load the Inner Ring is Rotating	Stationary	Single Row Bearings(2) $\frac{F_a}{VF_r} > e$		Double Row Bearings(3) $\frac{F_a}{VF_r} \leq e$		$\frac{F_a}{VF_r} > e$		e
			V	V	X	Y	X	Y	X	Y	
Radial Contact Groove Ball Bearings(4)	$\frac{F_a}{C_o}$ (1)	$\frac{F_a}{iZD^2}$ Units lb, in.									
	0.014	25	1	1.2	0.56	2.30	1	0	0.56	2.30	0.19
	0.028	50				1.99				1.99	0.22
	0.056	100				1.71				1.71	0.26
	0.084	150				1.55				1.55	0.28
	0.11	200				1.45				1.45	0.30
	0.17	300				1.31				1.31	0.34
	0.28	500				1.15				1.15	0.38
	0.42	750				1.04				1.04	0.42
	0.56	1000				1.00				1.00	0.44
Angular Contact Groove Ball Bearings with Contact Angle(4):	$\frac{iF_a}{C_o}$ (1)	$\frac{F_a}{ZD^2}$ Units lb, in.									
5°	0.014	25	1	1.2	For this type use the X, Y and e values applicable to single row radial contact bearings		1	2.78	0.78	3.74	0.23
	0.028	50						2.40		3.23	0.26
	0.056	100						2.07		2.78	0.30
	0.085	150						1.87		2.52	0.34
	0.11	200						1.75		2.36	0.36
	0.17	300						1.58		2.13	0.40
	0.28	500						1.39		1.87	0.45
	0.42	750						1.26		1.69	0.50
	0.56	1000						1.21		1.63	0.52
10°	0.014	25	1	1.2	0.46	1.88	1	2.18	0.75	3.06	0.29
	0.029	50				1.71		1.98		2.78	0.32
	0.057	100				1.52		1.76		2.47	0.36
	0.086	150				1.41		1.63		2.29	0.38
	0.11	200				1.34		1.55		2.18	0.40
	0.17	300				1.23		1.42		2.00	0.44
	0.29	500				1.10		1.27		1.79	0.49
	0.43	750				1.01		1.17		1.64	0.54
	0.57	1000				1.00		1.16		1.63	0.54
15°	0.015	25	1	1.2	0.44	1.47	1	1.65	0.72	2.39	0.38
	0.029	50				1.40		1.57		2.28	0.40
	0.058	100				1.30		1.46		2.11	0.43
	0.087	150				1.23		1.38		2.00	0.46
	0.12	200				1.19		1.34		1.93	0.47
	0.17	300				1.12		1.26		1.82	0.50
	0.29	500				1.02		1.14		1.66	0.55
	0.44	750				1.00		1.12		1.63	0.56
	0.58	1000				1.00		1.12		1.63	0.56
20°			1	1.2	0.43	1.00	1	1.09	0.70	1.63	0.57
25°			1	1.2	0.41	0.87	1	0.92	0.67	1.41	0.68
30°			1	1.2	0.39	0.76	1	0.78	0.63	1.24	0.80
35°			1	1.2	0.37	0.66	1	0.66	0.60	1.07	0.95
40°			1	1.2	0.35	0.57	1	0.55	0.57	0.93	1.14
Self-aligning Ball Bearings			1	1	0.40	0.4 cot α	1	0.42 cot α	0.65	0.65 cot α	1.5 tan α

At Section B-B, point Q:

(d) M = 800 in-lb

$$r_n = \frac{\left[r_o^{1/2} + r_i^{1/2}\right]^2}{4} = \frac{\left[(4.5)^{1/2} + (3.5)^{1/2}\right]^2}{4} = 3.984''$$

$$e = 4.0 - 3.984 = .016'', \qquad h_i = .5 - .016 = .484''$$

(e) Bending stress + direct tension $= \dfrac{Mh_i}{Aer_i} + \dfrac{P}{A} = \dfrac{800(.484)}{\frac{1}{4}\pi(1)^2(.016)(3.5)} + \dfrac{200}{\frac{1}{4}\pi(1)^2}$

$$= 8800 + 260 = 9060 \text{ psi tension}$$

(f) Maximum shear $= \frac{1}{2}(9060) = 4530$ psi at point Q.

4. An offset bar is loaded as shown in Fig. 4-5. The weight of the bar can be neglected. What is the maximum offset (dimension X) if the allowable stress in tension is limited to 10,000 psi? Where will the maximum tensile shear stress occur?

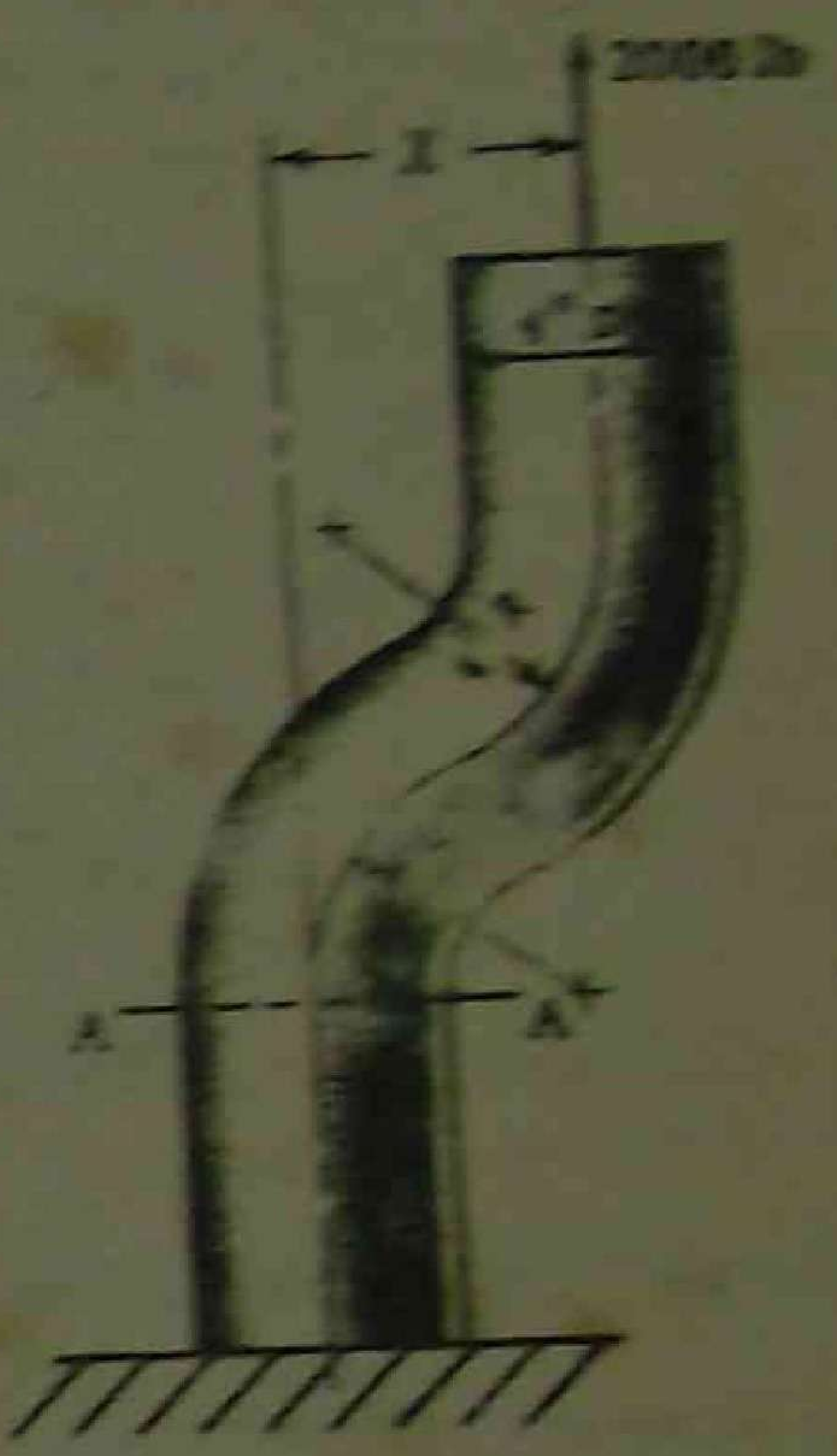


Fig. 4-5

Solution:

(a) Since the bar is symmetrical, bending causes the greatest stress at the inside fiber. Point P on section A-A will be stressed the greatest.

(b) M for section A-A = 2000(X) in-lb

$$r_n = \frac{\left[r_o^{1/2} + r_i^{1/2}\right]^2}{4} = \frac{\left[(6.0)^{1/2} + (2.0)^{1/2}\right]^2}{4} = 3.732''$$

$$e = R - r_n = 4 - 3.732 = .268''$$
$$h_i = 2 - .268 = 1.732''$$
$$A = \tfrac{1}{4}\pi(4)^2 = 12.56 \text{ in}^2$$

(c) The allowable tensile stress is 10,000 psi.

Bending tensile stress + direct tensile stress $= \dfrac{Mh_i}{Aer_i} + \dfrac{P}{A}$

or
$$10{,}000 = \frac{2000(X)(1.732)}{12.56(.268)(2)} + \frac{2000}{12.56}$$

from which $X = 19''$, the maximum offset.

5. Set up the basic relations necessary to obtain the stress distribution in a curved beam due to bending alone and derive the equation to give the bending stress distribution. See Fig. 4-6.

Solution:

(a) Consider a differential element of the beam subtending an angle $d\theta$.

(b) As a result of bending, and *plane sections remaining plane*, an arbitrary section p-q rotates to p′-q′, with tension on the inner fiber and compression on the outer fiber. The rotation occurs with a point on the neutral axis remaining fixed.

(c) The elongation of the fiber a distance y from the neutral surface is $y\,d\phi$.

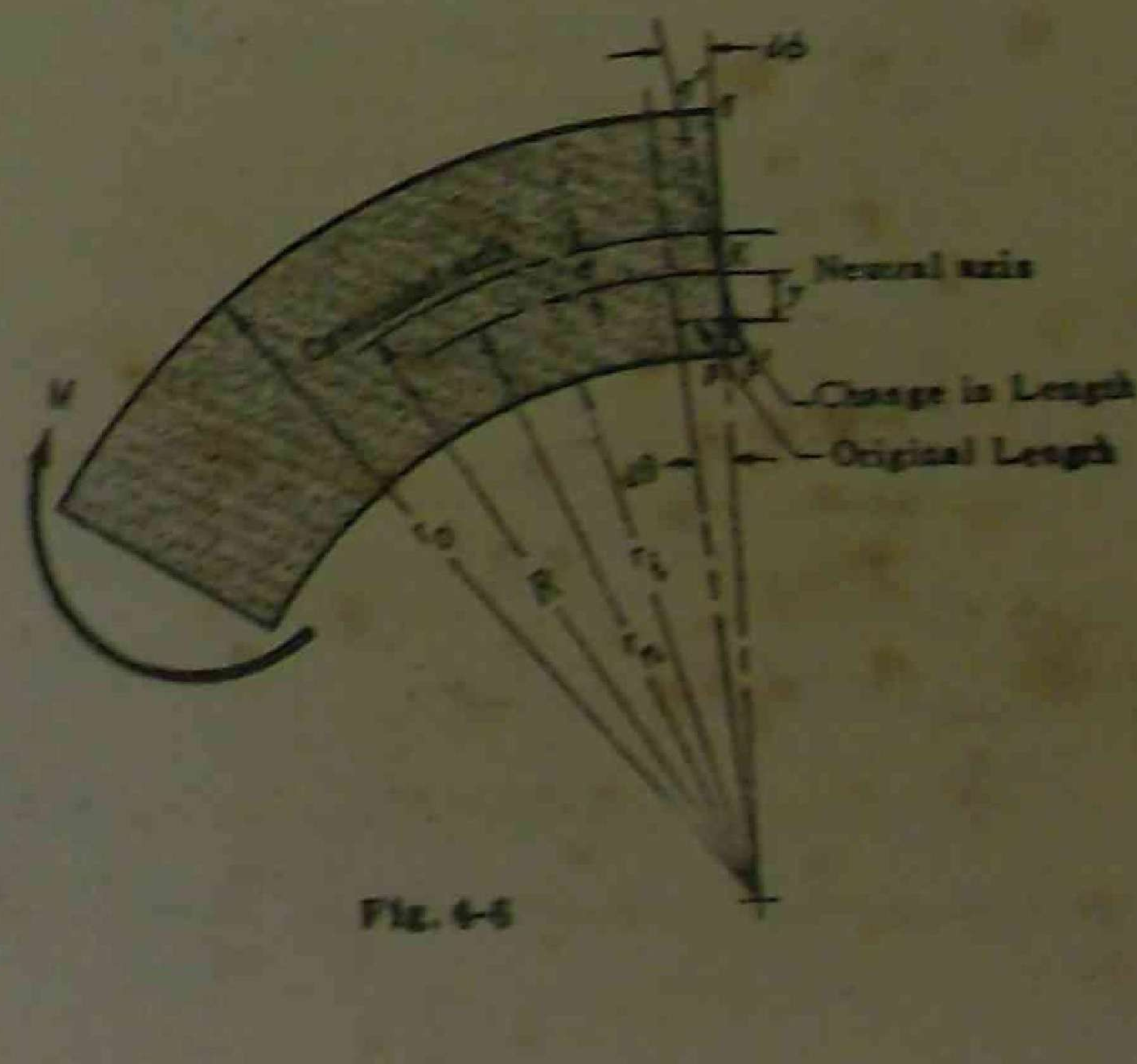


Fig. 4-6

(*d*) The original length of the differential fiber is $(r_n - y)\,d\theta$.

(*e*) Since stress is proportional to strain,

$$s = \epsilon E = \frac{\Delta l}{l}E \quad \text{or} \quad s = \frac{y\,d\phi}{(r_n - y)\,d\theta}E, \qquad \text{where } s \text{ is the bending stress.}$$

(*f*) The summation of all differential forces must be zero for equilibrium; hence

$$\int s\,dA = 0 \quad \text{or} \quad \int \frac{y\,d\phi\,E\,dA}{(r_n - y)\,d\theta} = 0 \quad \text{or} \quad \frac{d\phi}{d\theta}E\int\frac{y\,dA}{r_n - y} = 0$$

(*g*) Also, the moment of the differential forces about any point should be equal to the applied couple *M*. Take the point *K* as a convenient center of moments.

$$\int s\,y\,dA = M \quad \text{or} \quad \int\left[\frac{y\,d\phi}{(r_n - y)\,d\theta}E\right]y\,dA = M \quad \text{or} \quad \frac{d\phi}{d\theta}E\int\frac{y^2}{r_n - y}\,dA = M$$

(*h*) Manipulation of $\displaystyle\int\frac{y^2}{r_n - y}\,dA = M$, by dividing $(r_n - y)$ into y^2, gives

$$r_n\int\frac{y}{r_n - y}\,dA - \int y\,dA = M$$

(*i*) But from (*f*), $\displaystyle\int\frac{y}{r_n - y}\,dA = 0$ and $\int y\,dA$ represents the moment about the neutral axis of the differential areas comprising the section. Hence $\int y\,dA$ can be written as Ae, where e is the distance from the neutral axis to the centroidal axis.

(*j*) Thus the equation in (*g*) can be written $\displaystyle\frac{d\phi}{d\theta}E\int\frac{y^2}{r_n - y}\,dA = M = \frac{d\phi}{d\theta}E\,[Ae]$ or $\displaystyle\frac{d\phi}{d\theta}E = \frac{M}{Ae}$.

(*k*) The stress equation in (*e*) can be written $\displaystyle s = \frac{y\,d\phi}{(r_n - y)\,d\theta}E = \frac{M}{Ae}\,\frac{y}{r_n - y}$ which gives the stress variation.

6. A section of a C-clamp is shown in Fig. 4-7. What force *F* can be exerted by the screw if the maximum tensile stress in the clamp is limited to 20,000 psi?

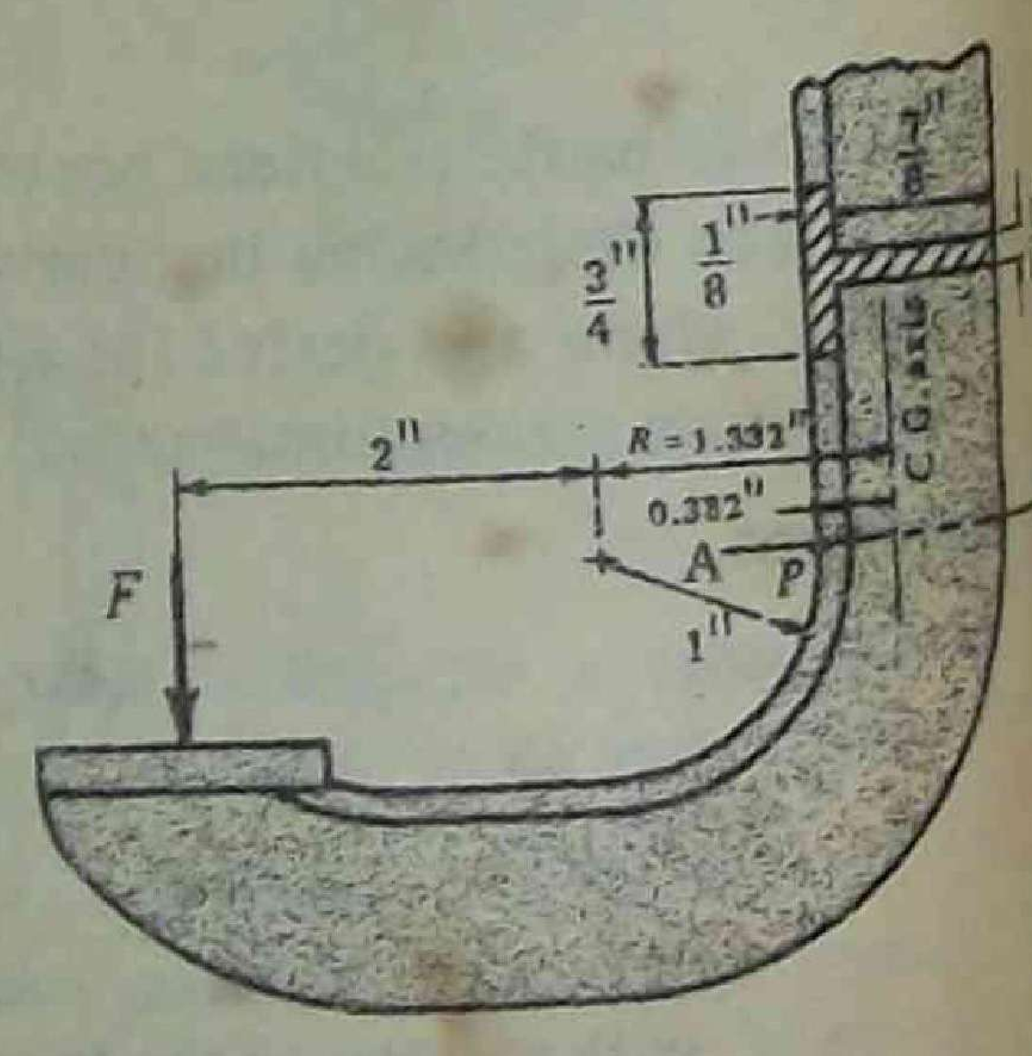


Fig. 4-7

Solution:

(*a*) The maximum tensile stress will occur at point *P* at section A-A, on which section the bending is maximum, curvature exists, and a direct tensile stress acts.

(*b*) The distance from the center of curvature to the C.G. axis is, from Table I,

$$R = r_i + \frac{\frac{1}{2}h^2t + \frac{1}{2}t_i^2(b_i - t)}{ht + (b_i - t)t_i} = 1 + \frac{\frac{1}{2}(1)^2(\frac{1}{8}) + \frac{1}{2}(\frac{1}{8})^2(\frac{3}{4} - \frac{1}{8})}{(1)(\frac{1}{8}) + (\frac{3}{4} - \frac{1}{8})(\frac{1}{8})}$$

$$= 1.332''$$

(*c*) Also, from Table I,

$$r_n = \frac{(b_i - t)(t_i) + th}{(b_i - t)\log_e\frac{r_i + t_i}{r_i} + t\log_e\frac{r_o}{r_i}} = \frac{(\frac{3}{4} - \frac{1}{8})(\frac{1}{8}) + (\frac{1}{8})(1)}{(\frac{3}{4} - \frac{1}{8})\log_e\frac{1 + \frac{1}{8}}{1} + \frac{1}{8}\log_e\frac{2}{1}}$$

$$= 1.267''$$

(d) $e = R - r_n = 1.332 - 1.267 = 0.065''$. $h_i = r_n - r_i = 1.267 - 1.0 = 0.267''$.

(e) Area $= \frac{1}{8}(\frac{3}{4}) + \frac{7}{8}(\frac{1}{8}) = 0.203$ sq in.

(f) Bending moment (about C.G.) $= F(2+1.332) = 3.332F$

(g) Bending stress + direct stress $= \dfrac{Mh_i}{Aer_i} + \dfrac{F}{A}$

$$20{,}000 = \frac{3.332F\,(0.267)}{(0.203)(0.065)(1)} + \frac{F}{0.203} \quad \text{and} \quad F\,(\text{maximum}) = 276 \text{ lb}$$

(h) Note that the stress at the outer fiber may be larger in this case than at the inner fiber, but this stress at the outer fiber is compression.

7. A trough 1"thick by 8"long is subjected to a concentrated load of 400 lb. Determine the magnitude and location of the maximum tension, compression, and shear stresses.

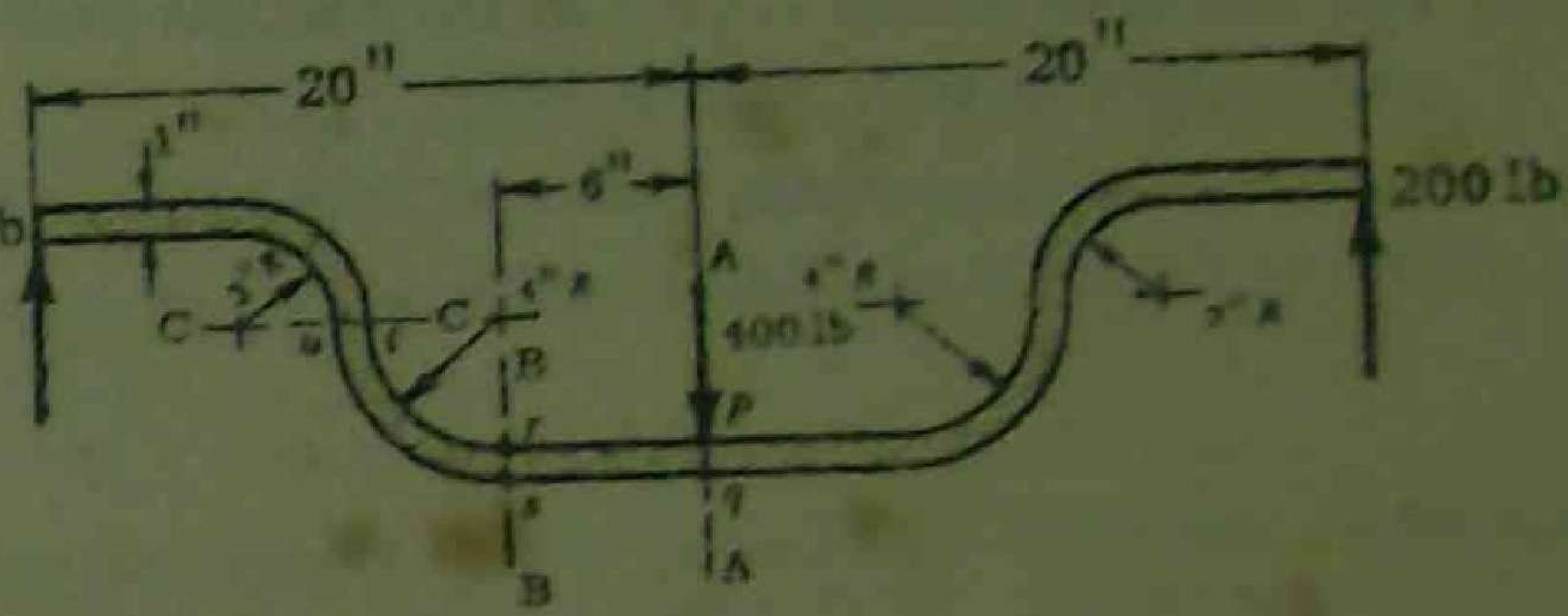


Fig. 4-8

Solution:

(a) The location of the maximum bending stress can be reduced, by inspection, to three different locations:

1) Section A-A
2) Section B-B
3) Section C-C

At section A-A, the bending moment is maximum but the beam is straight.

At section B-B, the bending moment is less than at A-A but the beam has curvature.

At section C-C, the bending moment is less than at A-A and B-B but the beam has a smaller radius of curvature. Also, at C-C a direct tensile stress is applied, which is not present at the other two sections.

Hence the stress will be computed at the three different sections and compared. (The stresses need be computed on only one side of the centerline since the beam is symmetrical.)

(b) <u>Section A-A</u>

$$s = \frac{Mc}{I} = \frac{Mc}{bh^3/12} = \frac{(200\times 20)\frac{1}{2}}{8(1^3)/12} = 3000 \text{ psi (tension at point } q\text{, compression at point } p)$$

Note. Transverse shear stress is zero at points p and q.

(c) <u>Section B-B</u>

<u>Point r:</u> $s_i = \dfrac{Mh_i}{Aer_i} = \dfrac{(200\times 14)(0.5 - 0.019)}{8(0.019)(4)} = 2215$ psi (compression)

where $r_n = \dfrac{h}{\log_e r_o/r_i} = \dfrac{1}{\log_e 5/4} = 4.481''$. $e = R - r_n = 4.5 - 4.481 = 0.019''$

<u>Point s:</u> $s_o = \dfrac{Mh_o}{Aer_o} = \dfrac{(200\times 14)(0.5 + 0.019)}{8(0.019)(5)} = 1910$ psi (tension)

(d) <u>Section C-C</u>

<u>Point t:</u> At point t the stress due to bending is compression and the direct stress is tension. Hence the

Notes: (1) C_0 is the static basic load rating.

(2) For single row bearings, when $\frac{F_a}{VF_r} \le e$, use $X = 1$ and $Y = 0$.

Two similar single row angular contact ball bearings mounted "face-to-face" or "back-to-back" are considered as one double row angular contact bearing.

For two or more similar single row ball bearings mounted "in tandem", use the values of X, Y and e which apply to one single row ball bearing. When α is smaller than 20°, F_r and F_a are not the total loads but the loads per single row bearing. C_0 and i also refer to one single row bearing.

(3) Double row bearings are presumed to be symmetrical.

(4) Permissible maximum value of F_a/C_0 depends on the bearing design.

(5) Values of X, Y and e for a load or contact angle other than shown in Table II-2 are obtained by linear interpolation.

(5) This standard is limited to bearings whose ring raceways have a cross sectional radius not larger than:

In deep groove and angular contact ball bearing inner rings:	52% of the ball diameter.
In deep groove and angular contact ball bearing outer rings:	53% of the ball diameter.
In self-aligning ball bearing inner rings:	53% of the ball diameter.

The basic load rating is not increased by the use of smaller groove radii, but reduced by the use of larger radii than those given above.

Appendix 1. A recommended value of the factor f based on current tests of ball bearings of good quality, hardened ball bearing steel is

$f = 10$ when kg and mm units are used
$f = 7450$ when pound and inch units are used.

III. METHOD OF EVALUATING DYNAMIC LOAD RATINGS FOR BALL BEARINGS HAVING CROSS SECTION RADII OF THE RING RACEWAYS ·.% OF THE BALL DIAMETER

A. Definitions are as given in II A above.

B. Calculation of Basic Load Rating, Rating Life and Equivalent Load

(1) It is recognized that revisions of this recommendation may be required from time to time as the result of improvements or new developments.

(2) Basic Load Rating: The magnitude of the basic load rating C, for radial and angular contact ball bearings, is

$$C = f_c (i \cos \alpha)^{0.7} Z^{2/3} D^{1.8}$$

where i, α, Z, D and f_c are as defined in II B above.

Values of f_c are obtained by multiplying the value of f_c/f from the appropriate column of Table III-1 by a factor f, covered in Appendix 1 of Section II.

(3) Rating Life: The approximate magnitude of the rating life L, of ball bearings, is

$$L = (C/P)^3 \text{ million revolutions}$$

where P = the equivalent load.

(4) Equivalent Load: The magnitude of the equivalent load P, for radial and angular contact ball bearings of conventional types, under combined constant radial and constant thrust loads, is

$$P = XVF_r + YF_a$$

where X = a radial factor
V = a rotation factor = 1.0 for inner ring rotating in relation to load
= 1.2 for inner ring stationary in relation to load
Y = a thrust factor, F_r = the radial load, F_a = the thrust load.

Values of X and Y are given in Table III-2. The factor V, due to lack of sufficient experimental evidence, is used as a matter of precaution.

Table III-1

Factor $\frac{f_c}{f}$

$\frac{D \cos \alpha}{d_m}$ (1)	$\frac{f_c}{f}$ Radial and angular contact groove ball bearings (2)
0.05	0.296
0.06	0.311
0.07	0.324
0.08	0.335
0.09	0.344
0.10	0.352
0.12	0.364
0.14	0.373
0.16	0.378
0.18	0.380
0.20	0.380
0.22	0.378
0.24	0.373
0.26	0.368
0.28	0.362
0.30	0.355
0.32	0.347
0.34	0.337
0.36	0.327
0.38	0.317
0.40	0.306

Notes (1) and (2) under Table II-1 apply also to Table III-1.

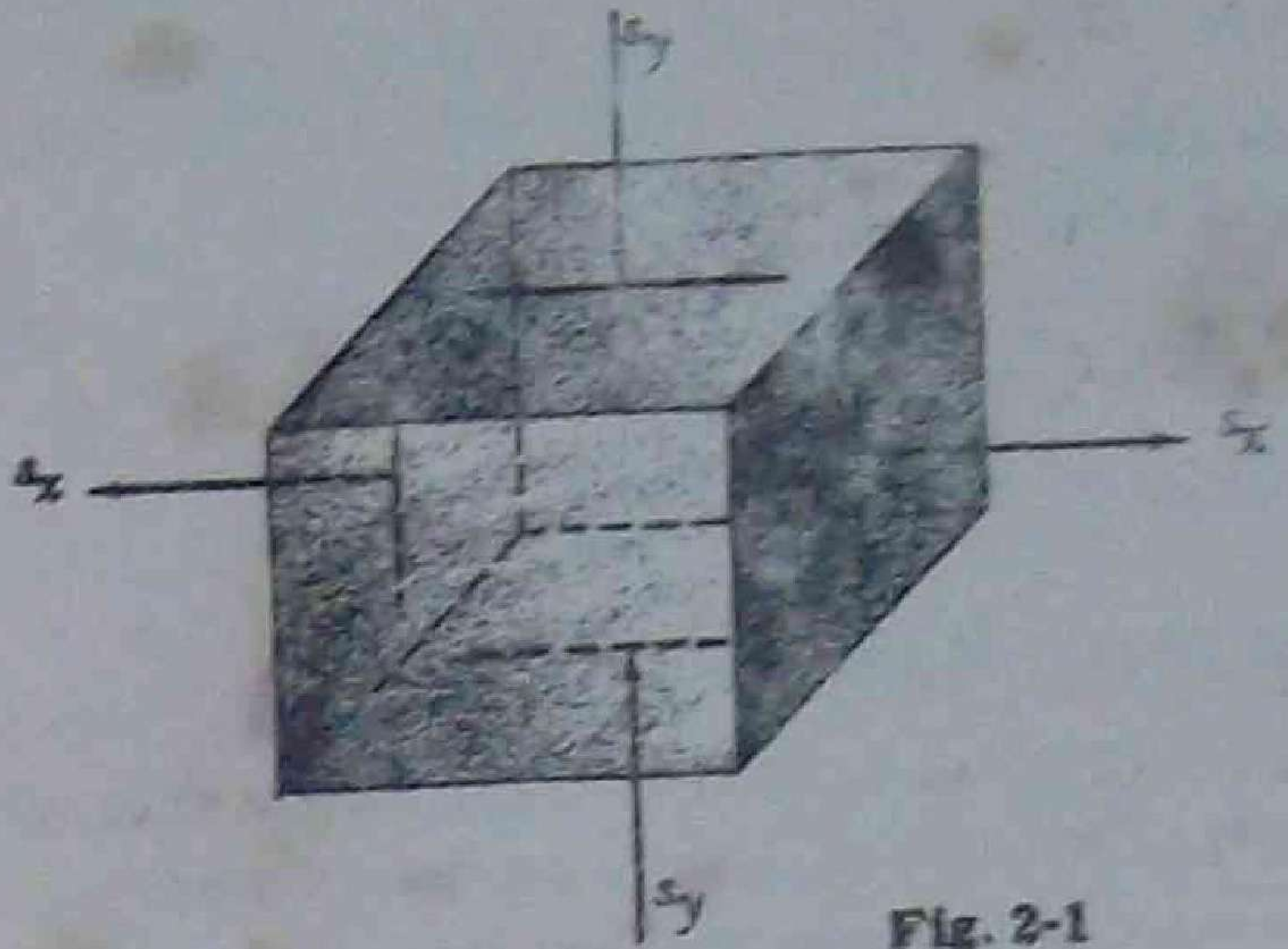


Fig. 2-1

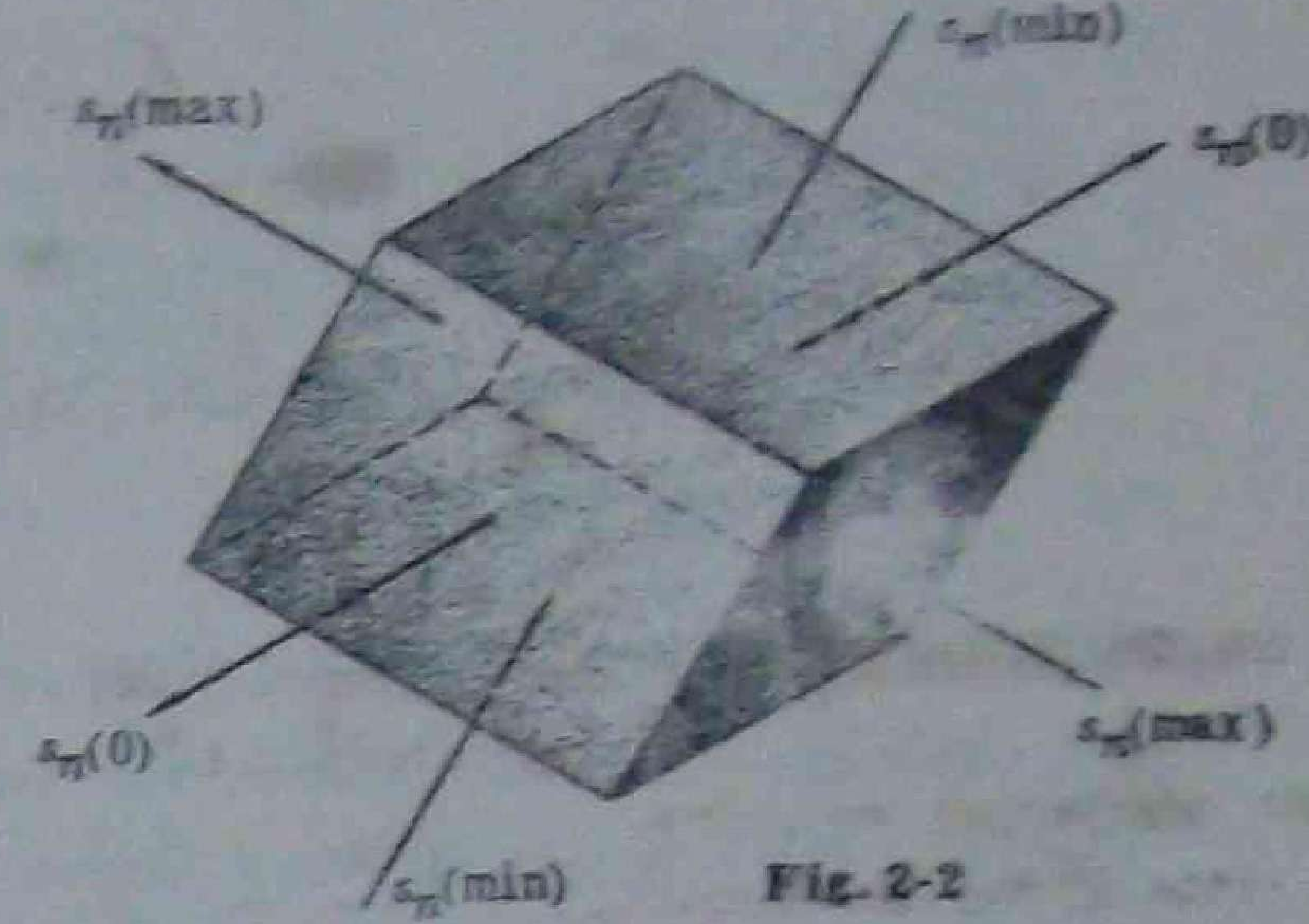


Fig. 2-2

THE MAXIMUM SHEAR STRESS, τ(max), at the critical point being investigated is equal to half of the greatest difference of any two of the three principal stresses (do not overlook any of the principal stresses which are zero). Hence, for the case of two-dimensional loading on a particle causing a two-dimensional stress,

$$\tau(\max) = \frac{s_n(\max) - s_n(\min)}{2} \quad \text{or} \quad \frac{s_n(\max) - 0}{2} \quad \text{or} \quad \frac{s_n(\min) - 0}{2}$$

depending upon which results in the greatest numerical value. The planes of maximum shear are inclined at 45° with the principal planes as shown in Fig. 2-3 below.

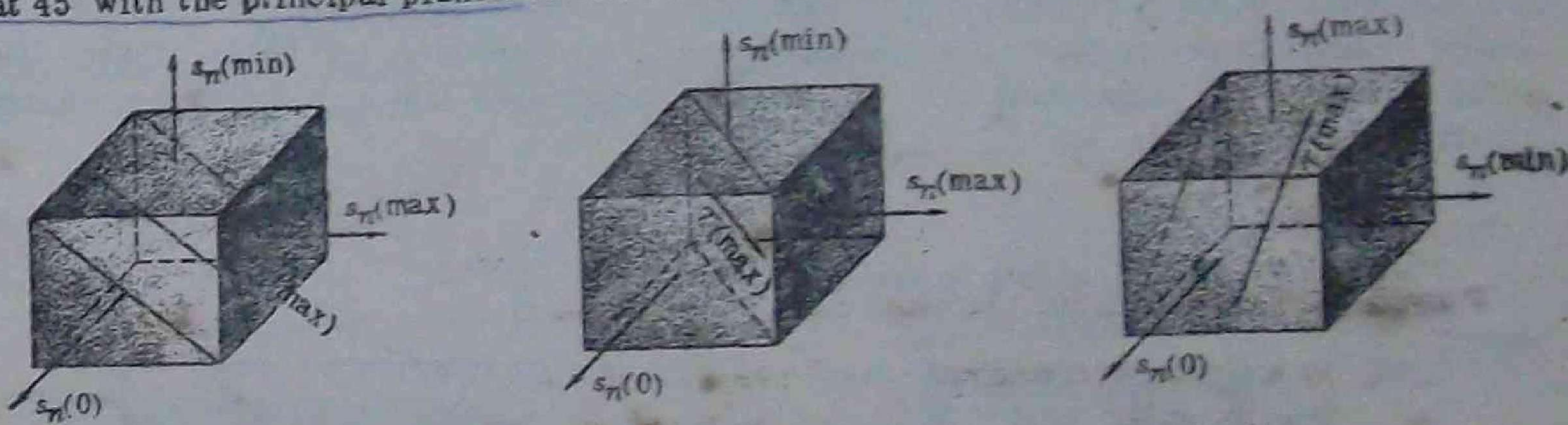


Fig. 2-3

THE APPLICATION of equations (*1*) and (*2*) requires the determination of s_x, s_y, and τ_{xy} at the critical point in the machine member. The critical point is the point at which the applied loads produce the maximum combined stress effects. In a beam, the following are representative stresses that can occur, to be included in equations (*1*) and (*2*) if they act at the same point.

s_x and $s_y = \pm \dfrac{Mc}{I} \pm \dfrac{P}{A}$, remembering that these stresses may be either plus or minus depending upon whether they are tension or compression.

$\tau_{xy} = \dfrac{Tr}{J} + s_v$ for a circular cross section (when these stresses are parallel).

M = bending moment, in-lb
c = distance from neutral axis to outer surface, in.
r = radius of circular cross section, in.
I = rectangular moment of inertia of cross section, in^4
P = axial load, lb
A = area of cross section, in^2
T = torsional moment, in-lb
J = polar moment of inertia of cross section, in^4
s_v = transverse shear, psi.

Table III-2

Factors X, Y and e

Bearing Type		Single Row Bearings (1) $\frac{F_a}{VF_r} > e$		Double Row Bearings (2) $\frac{F_a}{VF_r} \leq e$		Double Row Bearings (2) $\frac{F_a}{VF_r} > e$		e
		X	Y	X	Y	X	Y	
Radial Contact Groove Ball Bearings	$\frac{F_a}{iZD^2}$ Units lb, in.							
	25	0.56	3.09	1	0	0.56	3.09	0.09
	50		2.77				2.77	0.12
	100		2.43				2.43	0.14
	150		2.23				2.23	0.15
	200		2.10				2.10	0.16
	300		1.92				1.92	0.18
	500		1.71				1.71	0.21
	750		1.56				1.56	0.23
	1000		1.44				1.44	0.24
Angular Contact Groove Ball Bearings with Contact Angle:	$\frac{F_a}{ZD^2}$ Units lb, in.	For this type use the X, Y and e values applicable to single row radial contact bearings						
5°	25			1	3.89	0.78	5.02	0.17
	50				3.30		4.49	0.19
	100				2.89		3.94	0.22
	150				2.66		3.63	0.24
	200				2.50		3.41	0.25
	300				2.29		3.12	0.27
	500				2.04		2.78	0.31
	750				1.86		2.53	0.34
	1000				1.72		2.35	0.36
10°	25	0.46	2.20	1	2.55	0.75	3.58	0.25
	50		2.09		2.41		3.29	0.26
	100		1.94		2.24		3.14	0.28
	150		1.84		2.13		2.99	0.29
	200		1.77		2.04		2.87	0.31
	300		1.66		1.92		2.69	0.33
	500		1.53		1.77		2.49	0.35
	750		1.44		1.66		2.33	0.38
	1000		1.36		1.57		2.21	0.40
15°	25	0.44	1.55	1	1.74	0.72	2.52	0.35
	50		1.51		1.70		2.46	0.36
	100		1.48		1.66		2.41	0.36
	150		1.42		1.59		2.31	0.38
	200		1.39		1.56		2.25	0.39
	300		1.34		1.50		2.17	0.41
	500		1.26		1.42		2.05	0.43
	750		1.20		1.35		1.96	0.45
	1000		1.16		1.30		1.88	0.47
20°		0.43	1.14	1	1.25	0.75	1.86	0.50
25°		0.41	0.99	1	1.06	0.67	1.55	0.62
30°		0.39	0.81	1	0.83	0.63	1.31	0.75
35°		0.37	0.69	1	0.69	0.60	1.12	0.91
40°		0.35	0.66	1	0.54	0.57	0.97	1.08

... is used where protection against dirt is required on one side and where oil is available, usually by splash feed, on the other.

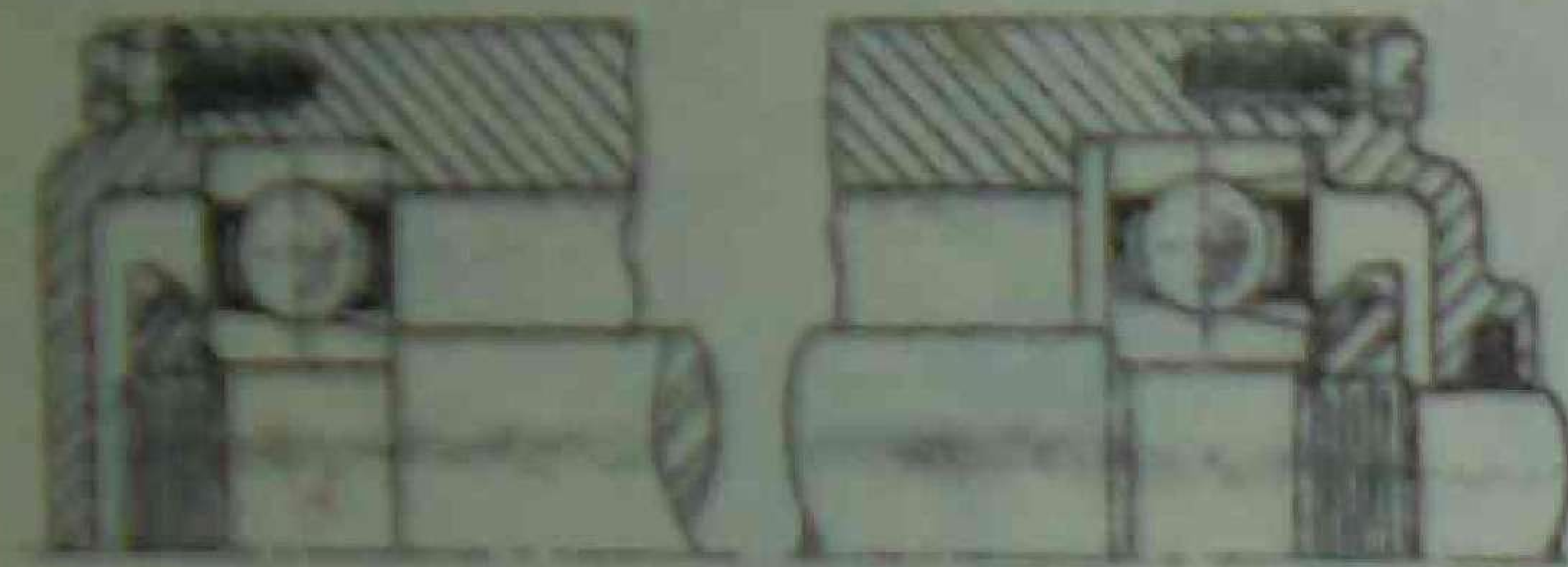


Fig. 22-3

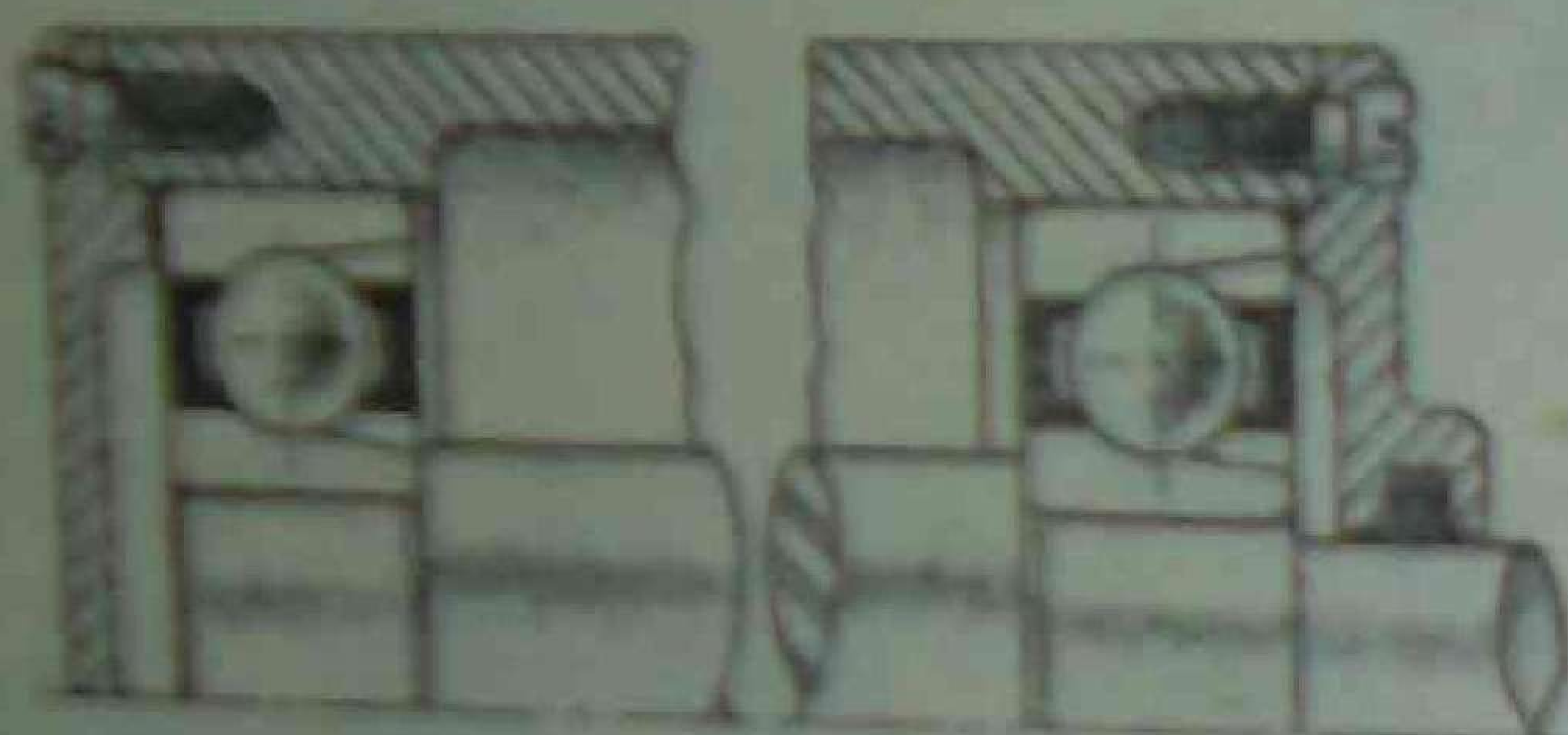


Fig. 22-4

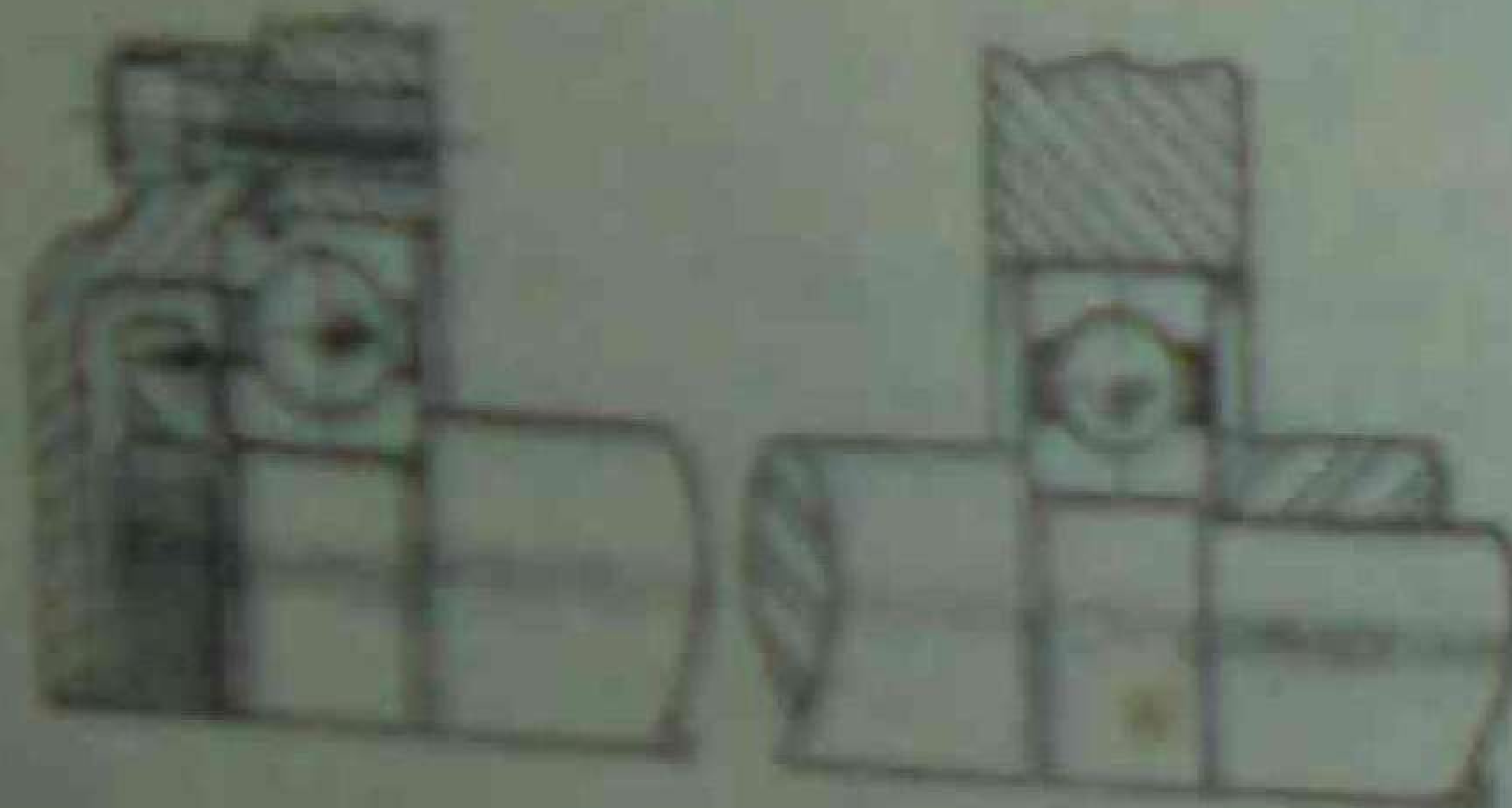


Fig. 22-5

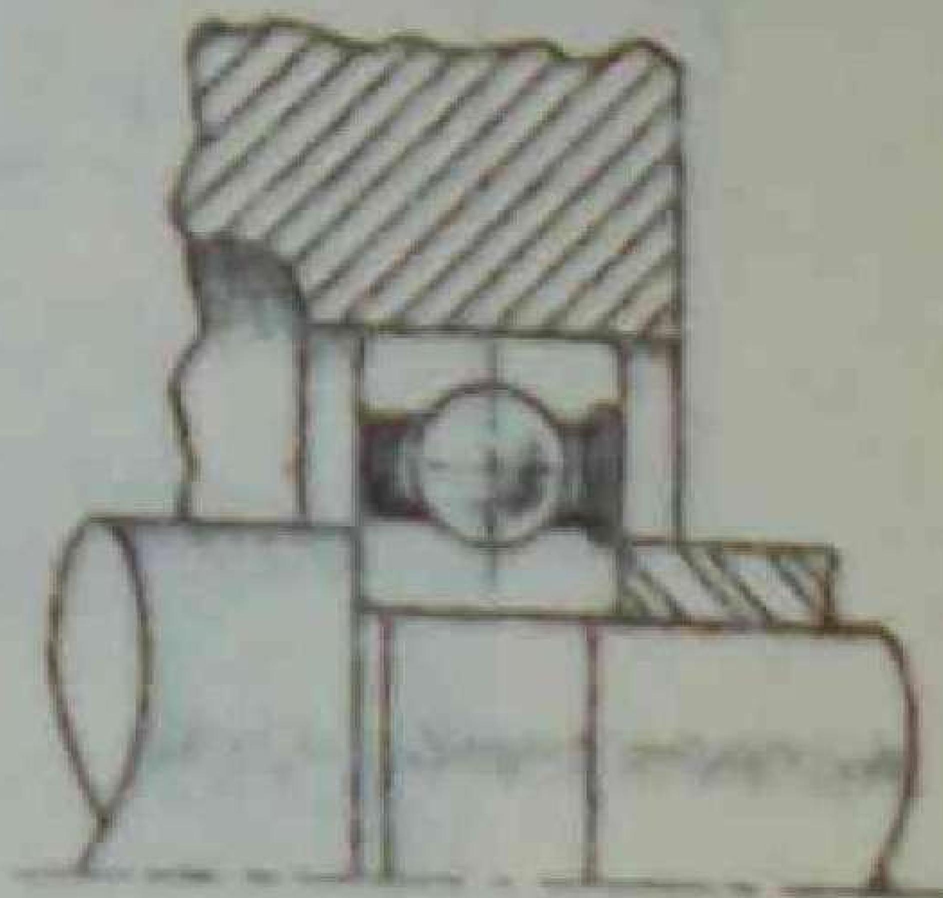


Fig. 22-6

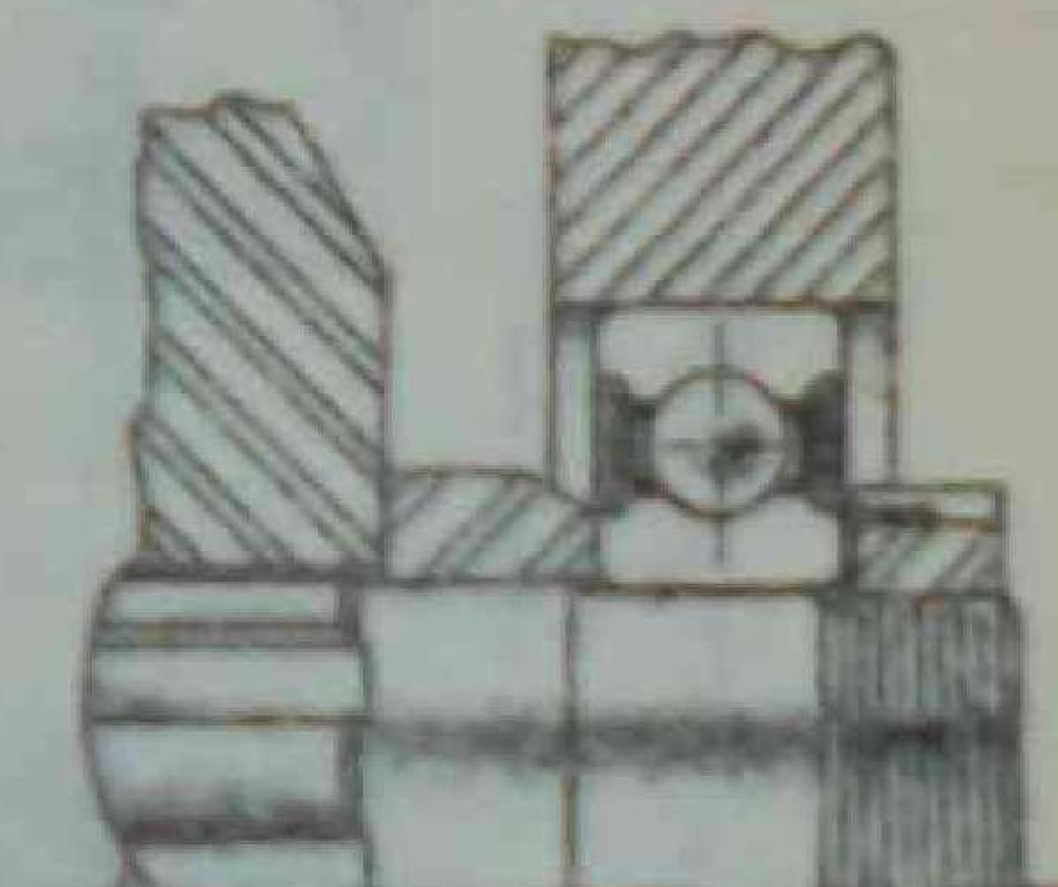


Fig. 22-7

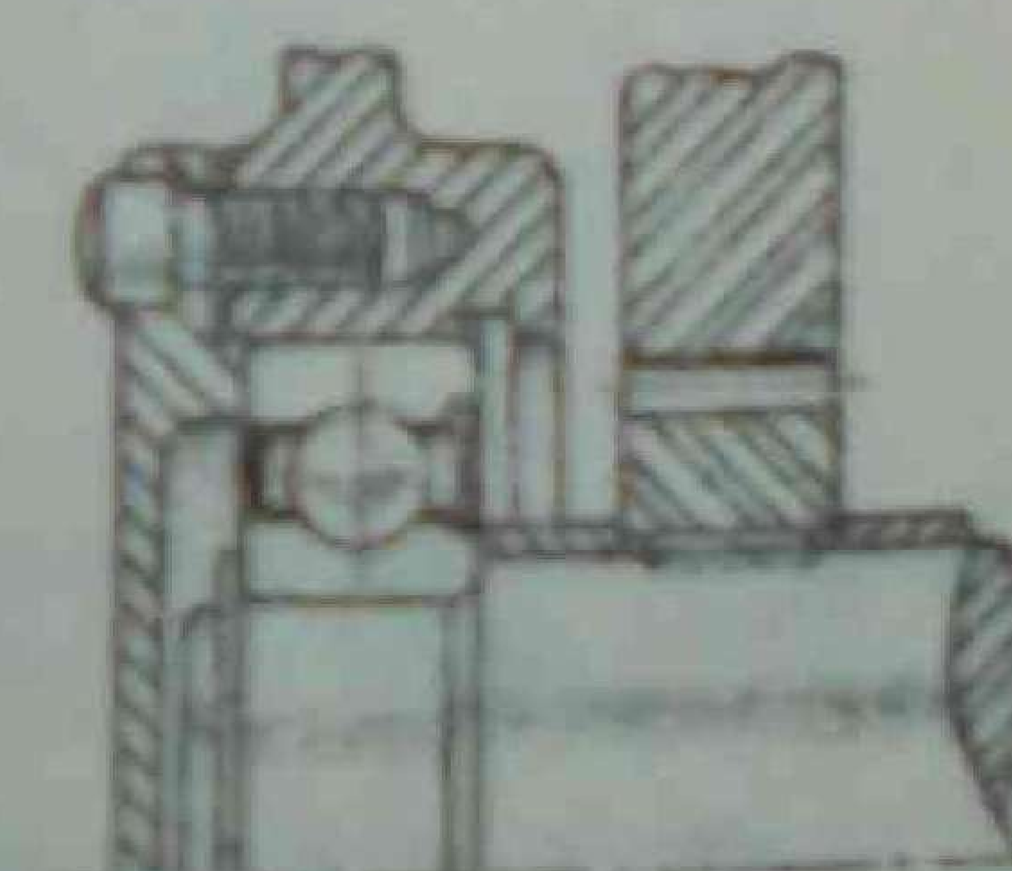


Fig. 22-8

Courtesy New Departure

A bearing with a shield and seal ... bearing

1. What is the approximate friction horsepower loss in a single radial ball bearing having a bore diameter of 2.1654" and subjected to a radial load of 5000 lb? The shaft rotates at 600 rpm.

Solution:

Friction torque $M_t = Ff(D/2) = (5000)(0.0015)(2.1654/2) = 8.1$ in-lb

Friction hp loss $= M_t N/63{,}000 = (8.1)(600)/63{,}000 = 0.077$ hp

2. Determine the approximate friction torque M_t expected in a radial deep groove bearing under a radial load of 1960 lb. The bearing is series 302, with a bore of 0.5906"–0.5903".

Solution:

Friction torque $M_t = Ff(D/2) = (1960)(0.0015)(0.5906/2) = 0.868$ in-lb

3. Derive Stribeck's equation for the static capacity of a single row radial deep groove ball bearing, assuming rigid races and equally spaced balls. Also, determine the maximum load on a ball. (Express the result in terms of the diameter D of the balls and the number of balls, Z.)

Solution:

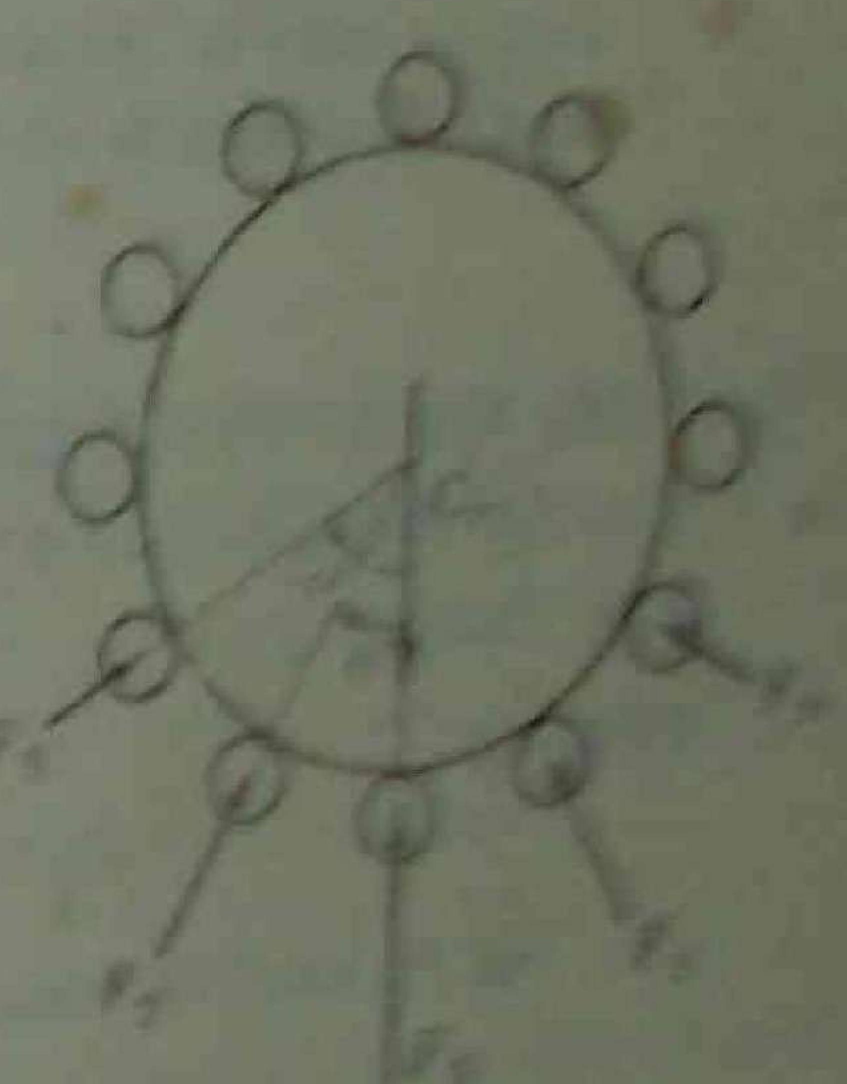


Fig. 22-9

(a) The radial load C_0 is balanced by the vertical components of the forces acting on the race through the balls in the lower half of the bearing:

$$C_0 = F_1 + 2F_2 \cos\theta + 2F_3 \cos 2\theta + \cdots$$

(b) A second consideration to permit solution of the above is obtained from deflection relations. The radial deflection at load F_1 is δ_1, that at load F_2 is δ_2, etc., with

$$\delta_2 = \delta_1 \cos\theta, \quad \delta_3 = \delta_1 \cos 2\theta \quad \text{etc.}$$

if the races are assumed to remain circular in shape.

(c) Also, the relation of deflections and loads is given by the following which are verified from the Hertz stress equations:

$$\frac{F_1}{F_2} = \frac{\delta_1^{3/2}}{\delta_2^{3/2}}, \quad \frac{F_1}{F_3} = \frac{\delta_1^{3/2}}{\delta_3^{3/2}} \quad \text{etc.}$$

(d) Substitution of (b) and (c) into (a) gives

$$C_0 = F_1[1 + 2(\cos\theta)^{5/2} + 2(\cos 2\theta)^{5/2} + \cdots]$$

(e) The angle θ depends upon the number of balls Z: $\theta = 360°/Z$.

(f) Rewrite (d) as $C_0 = F_1 M$, where $M = [1 + 2(\cos\theta)^{5/2} + 2(\cos 2\theta)^{5/2} + \cdots]$.

(g) Stribeck found that Z/M was practically a constant quantity regardless of the number of balls, the average value being about 4.37. He suggested using a value of 5 for practical conditions to account for internal clearance and out-of-round deformations that occurred. Later experimental work confirmed his conclusion. Thus the maximum load on a ball can be expressed, for a radial bearing under radial loading, as

$$F_1 = \frac{C_0 Z}{ZM} = \frac{C_0(5)}{Z}$$

(h) Stribeck found from experimental work that the load F_1 to produce a given permanent deformation between two balls of the same diameter could be given by

$$F_1 = KD^2$$

where K is a proportionality constant. Then $KD^2 = 5C_0/Z$ [illegible] is the static radial capacity of a bearing is

$$C_0 = KZD^2/5$$

[illegible] $C_0 = KZD^2/5$, and the maximum load on a ball is $F_1 = 5C_0/Z$.

17. A load varies continuously in magnitude in a sinusoidal manner. The direction remains fixed. For a total life of 20,000,000 revolutions at a speed of 400 rpm, determine the mean cubic load F_m if the maximum load is 1000 lb.

Solution:

Since the load variation is repetitive, the mean cubic load for one cycle will be the same as for every cycle. Therefore consider one cycle, or 1 revolution.

The load after any part of a revolution is given by $F = +500 - 500 \cos 2\pi N$, where N is the fraction of a revolution (when $N = 0$, $F = 0$; when $N = 1/2$ rev, $F = +1000$; when $N = 1$ rev, $F = 0$).

$$F_m = \sqrt[3]{\frac{\int_0^{L_n} F^3\, dN}{L_n}} = \sqrt[3]{\frac{\int_0^1 (500 - 500\cos 2\pi N)^3\, dN}{1}}$$

$$= \sqrt[3]{(500)^3\left[N - \frac{3\sin 2\pi N}{2\pi} + \frac{3}{2\pi}\left(\pi N + \frac{\sin 4\pi N}{4}\right) - \frac{\sin 2\pi N}{6\pi}(\cos^2 2\pi N + 2)\right]_0^1}$$

$$= \sqrt[3]{(500)^3(2.5)} = 679 \text{ lb}$$

Bearing catalogs state that for the case where the load varies as a sine curve, the cubic mean load is obtained by the approximate formula of $F_m = 0.68\, F_{max}$, which agrees with the above calculation.

18. A shaft rotating at constant speed has variable load applied to it. The radial load on a bearing is $F_1 = 500$ lb for $t_1 = 1$ second, $F_2 = 300$ lb for $t_2 = 2$ seconds, $F_3 = 100$ lb for $t_3 = 3$ seconds. The load variation then repeats itself. What is the equivalent load F_m?

Solution:

$$F_m = \sqrt[3]{\frac{\Sigma F^3 t}{T}} = \sqrt[3]{\frac{F_1^3 t_1 + F_2^3 t_2 + F_3^3 t_3}{T}} = \sqrt[3]{\frac{(500)^3(1) + (300)^3(2) + (100)^3(3)}{6}} = 312 \text{ lb}$$

Note. The equation $F_m = \sqrt[3]{\dfrac{\Sigma F^3 N}{L_n}} = \sqrt[3]{\dfrac{F_1^3 N_1 + F_2^3 N_2 + F_3^3 N_3}{L_n}}$ could also be used here with the same final result. The revolutions for loads F_1, F_2 and F_3, for H hours of operation, are respectively $N_1 = \frac{1}{6}(60H)(\text{rpm})$, $N_2 = \frac{2}{6}(60H)(\text{rpm})$, $N_3 = \frac{3}{6}(60H)(\text{rpm})$; and the total revolutions $L_n = (60H)(\text{rpm})$. Substituting the values in the above equation, we obtain $F_m = 312$ lb.

19. The shaft shown in Fig. 22-11 below has mounted on it a spur gear G and a pulley P. Power is supplied to the pulley by means of a flat belt; power is taken from the shaft through the gear. The shaft is supported by two deep groove bearings. The following information has been established:

- Horsepower = 10 (steady load conditions)
- Speed of shaft = 900 rpm
- Shaft to be machined from hot rolled AISI 1035
- Diameter of the pulley = 10.0"
- Pitch diameter of the gear = 10.0"
- Weight of the pulley is approximately 30 lb
- Weight of the gear is approximately 30 lb
- Ratio of belt tensions $T_1/T_2 = 2.5$
- Gear pressure angle = 20°
- The pulley and gear are assembled with light press fits and keyed to the shaft.

The belt forces are perpendicular to the paper with the tight side being T_1 and the slack side being T_2. The tangential force on the gear is F_t and the separating force is F_r. F_t is perpendicular to the paper.

The design of the shaft for strength, critical speed, and rigidity is discussed in the chapters on variable stresses, shaft design, and deflections. The selection of proper single row, deep groove ball bearings is to be made now.

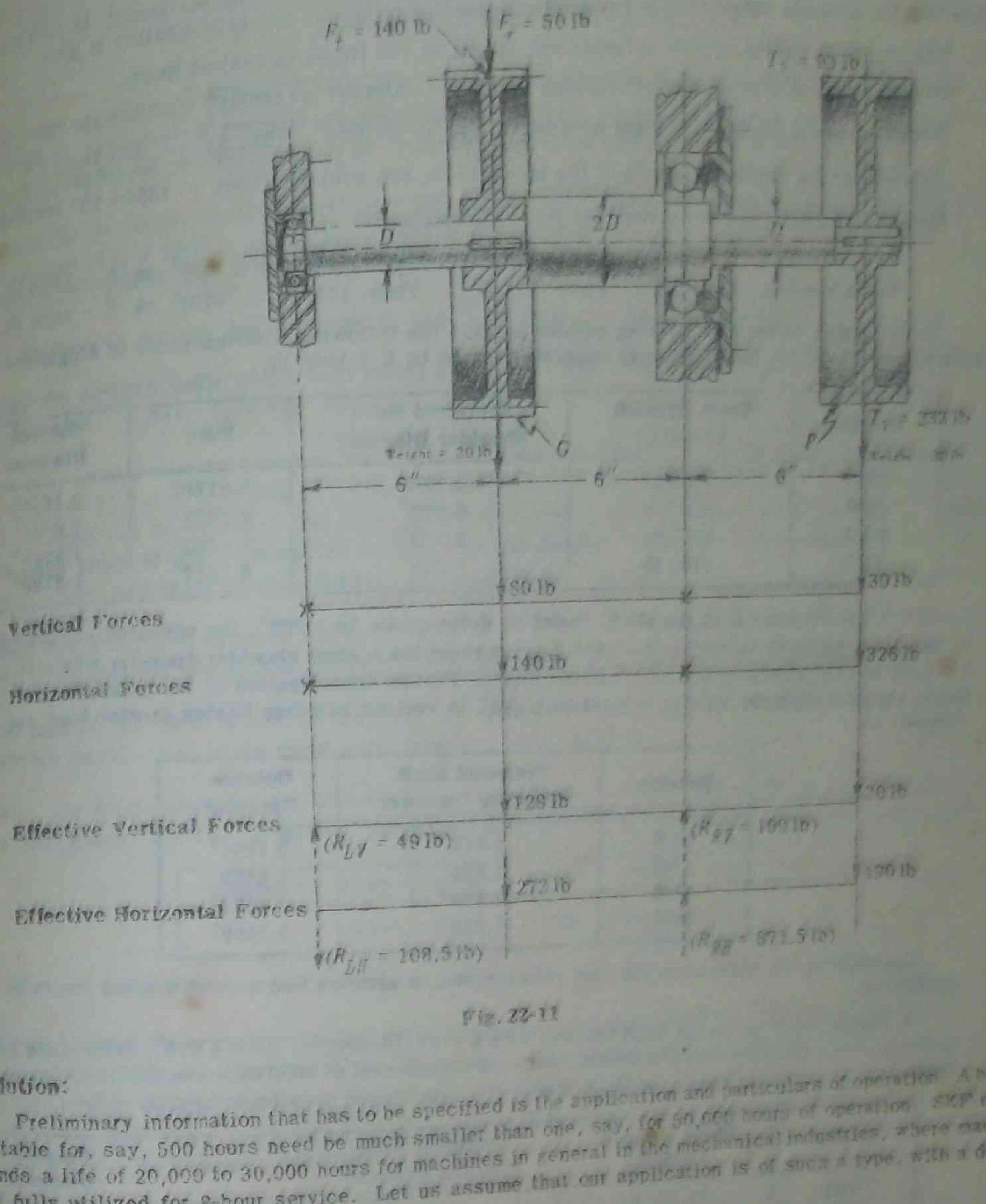


Fig. 22-11

Solution:

Preliminary information that has to be specified is the application and particulars of operation. A bearing suitable for, say, 500 hours need be much smaller than one, say, for 50,000 hours of operation. SKF recommends a life of 20,000 to 30,000 hours for machines in general in the mechanical industries, where machines are fully utilized for 8-hour service. Let us assume that our application is of such a type, with a desired life of 25,000 hours.

The dynamic effect in gear drives is due to two dynamic effects:

(1) the vibration introduced by inaccuracies in the gear tooth form, f_k

(2) the dynamic effect of the driven machine, f_d

The gear force is found from $$F_{eff} = F f_k f_d$$

where F is the theoretical load calculated from the torque and geometry. Values of f_k recommended by SKF range from 1.0 to 3.0. Let us arbitrarily use f_k = 1.3 and f_d = 1.5. Thus, $F_{eff} = F(1.3)(1.5) = 1.95F$. Values of f_d recommended by SKF range from 1.0 to 1.3, and values of ...

For V-belt drives, a factor f_b to take care of both the dynamic effect of belt vibration and the additional force necessary to maintain the proper tension in the belt varies from 1.5 to 2. Let us use f_b = 1.5 for the belt. Thus the effective belt force is $1.5F$.

The forces to be used in the calculations are:

Effective tangential force on the gear	= 1.95(140)	= 273 lb
Effective radial force on the gear	= 1.95(50)	= 98 lb
Effective belt force T_1	= 1.5(232)	= 348 lb
Effective belt force T_2	= 1.5(93)	= 140 lb

The effective sum of the belt tensions, $T_1 + T_2$, with the belt strands horizontal, is $350 + 140 = 490$ lb. Note that the dynamic effects are not considered applied to the weight of the pulley or gear.

The reactions for the effective forces are shown on the figure in dashed lines.

Resultant radial load on the left bearing is $R_L = \sqrt{(49)^2 + (108.5)^2} = 119$ lb.

Resultant radial load on the right bearing is $R_R = \sqrt{(109)^2 + (871.5)^2} = 879$ lb.

Revolutions L_n required for 90% of the bearings $= (25{,}000)(60)(900) = 1350 \times 10^6$ revolutions.

Specific dynamic capacity C required for each bearing is:

Left bearing: $L = (C/P)^3$ where $P = R_L$. Then $1350 = (C/119)^3$ or $C = 1300$ lb.

Right bearing: $L = (C/P)^3$ where $P = R_R$. Then $1350 = (C/879)^3$ or $C = 9650$ lb.

Investigation of the SKF bearing catalog reveals the following minimum sizes of single row deep groove bearings which have a basic dynamic capacity closest to $C = 1300$ lb:

Bearing	Basic Dynamic Capacity	Preferred Shaft Shoulder Diameter	Bore	Outside Diameter
6907	1690 lb	1.578"	1.3780"	2.1654"
6004	1620 lb	0.890"	0.7874"	1.6535"
6202	1320 lb	0.703"	0.6693"	1.5748"
6300	1400 lb	0.563"	0.3937"	1.3780"

Since the diameter D of the shaft, based on deflections, is 1.288", the preferred shoulder diameter can be used as a basis of selection, the 6907 bearing requiring a shaft shoulder diameter which is too large and the 6004 bearing requiring one which is too small. Further investigation of the catalog shows that the preferred shoulder diameter coming closest to 1.288" in various bearings having greater load capacities than required are:

Bearing	Preferred Shaft Shoulder Diameter	Outside Diameter
6006	1.346"	2.1654"
6206	1.406"	2.4409"
6305	1.469"	2.4409"
6405	1.339"	3.1496"

Considering the economics and size requirements, it appears that a 6006 bearing would be best for the application.

A similar analysis for the right bearing, with a shaft diameter of $2D = 2.576$", determined from a rigidity analysis, and a minimum specific dynamic capacity requirement of 9650 lb, gives the following minimum sizes in the various series of single row deep groove bearings, based upon load capacity only:

Bearing	Basic Dynamic Capacity	Preferred Shaft Shoulder Diameter	Bore	Outside Diameter
6018	11,500 lb	3.898"	3.5433"	5.5118"
6216	10,000 lb	3.504"	3.1496"	5.5118"
6310	10,700 lb	2.362"	1.9865"	4.3307"
6408	11,000 lb	1.969"	1.5478"	4.3307"

Examination of the catalog gives the minimum size bearing, shown in the adjacent table, coming closest to the required shaft shoulder diameter of $2D = 2.576$" (and having a specific dynamic capacity larger than 9650 lb. From an economic and size requirement, the 6311 bearing is suitable.

Bearing	Preferred Shaft Shoulder Diameter	Outside Diameter
6311	*2.559"	4.7244"
6411	2.638"	5.039"

*Slightly smaller than 2.576", but satisfactory.

The diameters as calculated for the different considerations are:

	Diameter D	Diameter $2D$
Strength: Soderberg equation (design factor 1.5)	$1\frac{1}{8}$"	$2\frac{1}{4}$"
ASME shafting code	1.01"	2.02"
Critical Speed (not considering transverse shear deflection)	0.597"	1.194"
Maximum slope of 1° at either bearing	0.470"	0.940"
Maximum deflection of 0.001" at the gear	1.288"	2.576"

relations, which ... shown.

JOURNAL BEARINGS make use of the basic theory of the converging film in order to support loads on a film of lubricant. Fig.23-3 below shows the end view of a journal bearing for the three positions of "rest", "start", and "run". Note that in the "rest" and "start" positions

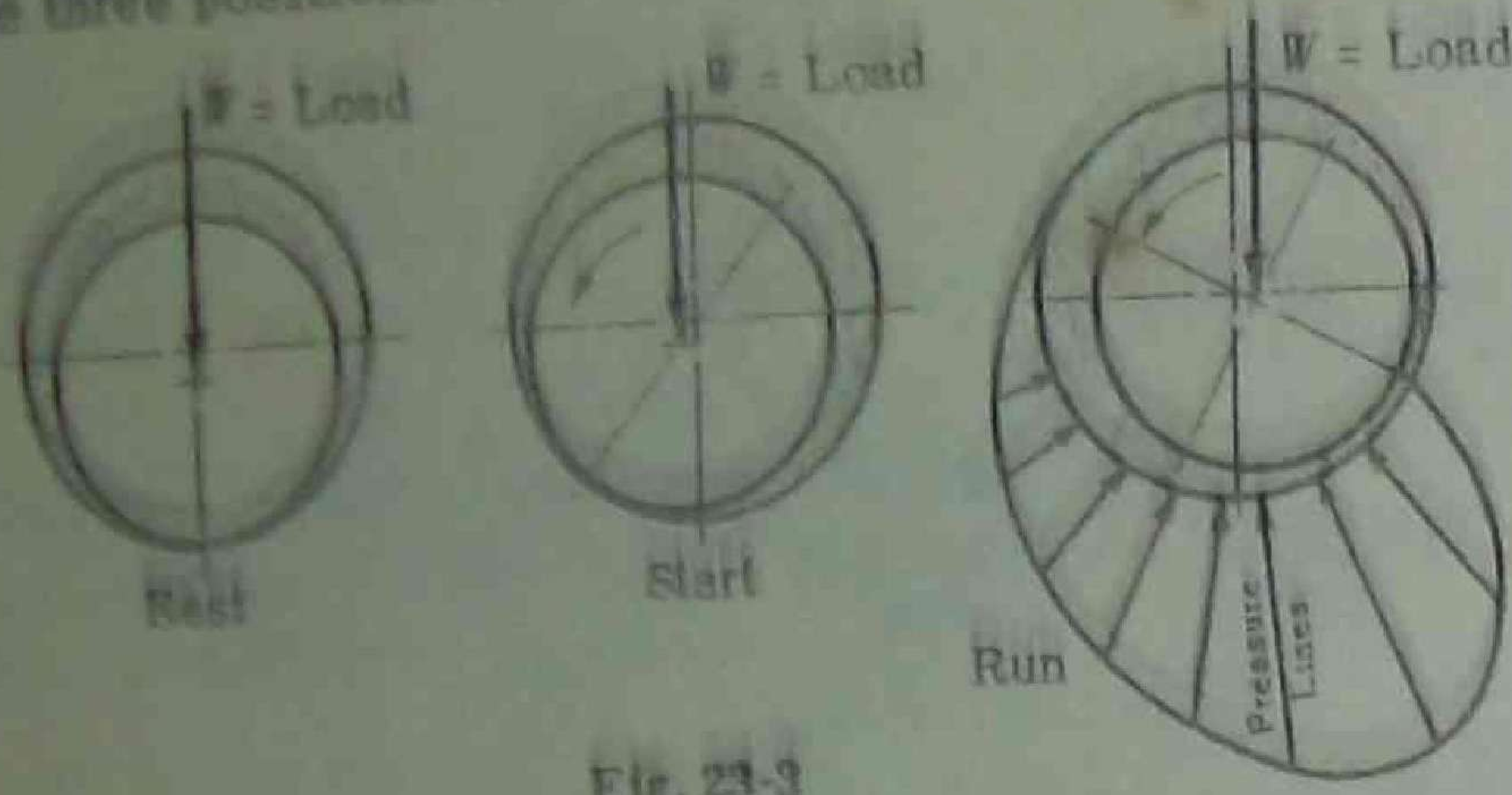


Fig. 23-3

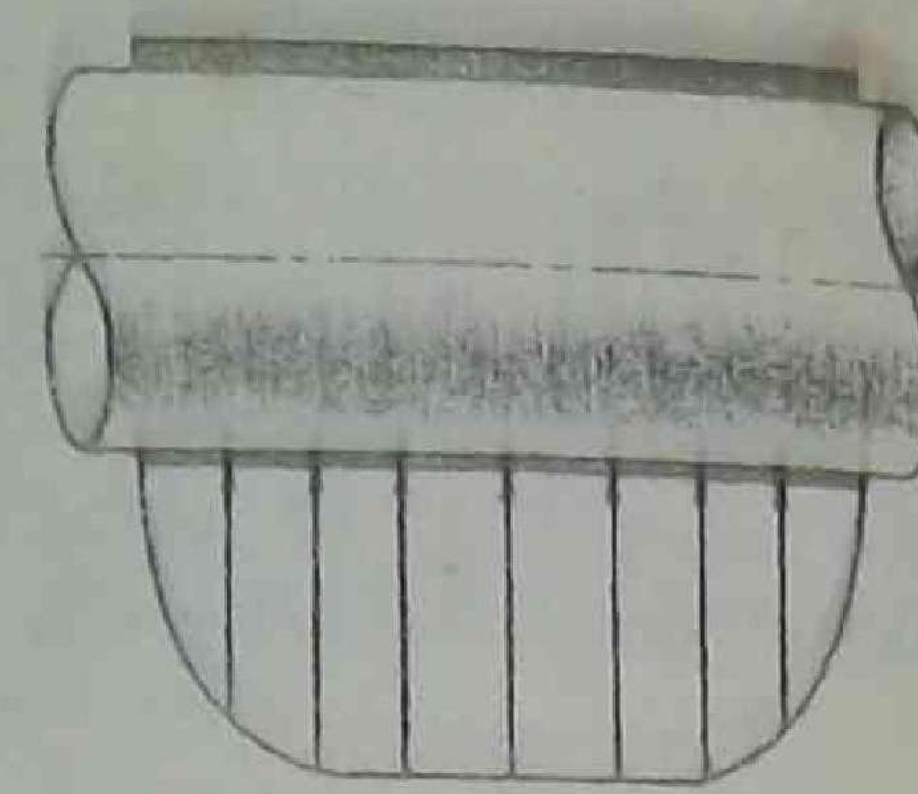


Fig. 23-4

there is contact between the journal (shaft) and the bearing (outer member). However, if the required conditions, as explained later, for perfect or thick-film lubrication are met, the shaft will be separated from the bearing by a film of lubricant as indicated in the "run" position, and the load will be supported by the film pressure. The terms thin-film or imperfect lubrication apply to the situation where bearing design and selection of lubricant have not met all the requirements for thick-film or perfect lubrication, and contact between journal and bearing is not completly prevented. Due to leakage of the lubricant from the ends of the bearing, there is a distribution of pressure in the axial direction as shown in Fig.23-4 above. The load carrying capacity of a journal bearing with perfect lubrication is a function of many variables, but essentially it involves the selection of the proper lubricant to provide perfect lubrication for specified operating conditions, and at the same time to provide for the proper heat balance between the heat generated within the bearing and the heat dissipated in order that the bearing will not exceed a safe specified operating temperature.

HEAT GENERATED, H_g, within a journal bearing is a function of the journal coefficient of friction f,

$$H_g = fW \frac{\pi DN}{12} \text{ ft-lb/min}$$

where H_g = heat generated, ft-lb/min
f = journal coefficient of friction
D = journal diameter, in.
N = journal speed, rpm
W = total radial bearing load, lb.

The main problem at this point is to be able to determine as closely as possible the value of the coefficient of journal friction. It is difficult to obtain a precise value for f, since it varies widely with operating conditions. The discussion in this chapter will be limited to full (360°) journal bearings.

Various investigators, employing dimensional analysis, have shown that the journal coefficient of friction is a function of at least three dimensionless parameters,

$$ZN/p, \qquad D/C, \qquad \text{and} \qquad L/D$$

where Z = absolute viscosity of lubricant at its operating temperature, centipoise
N = speed of journal, rpm; N' = speed of journal, rps (for later use)
p = bearing pressure based on projected area, W/LD psi
W = radial bearing load, lb
D = journal diameter, in.
C = diametral clearance between journal and bearing, in.
L = length of bearing, in.

The relationship between the coefficient of friction and the parameter ZN/p, called the bearing modulus, is of particular interest. The curve of Fig.23-5 is typical, but the slope and intercept of the straight line portion in the thick-film region depend upon variables such as the clearance ratio C/D, and the L/D ratio. Experimental data on small journal bearings by McKee established the following approximate equation for the coefficient of friction,

$$f = \frac{473}{10^{10}}\left(\frac{ZN}{p}\right)\frac{D}{C} + k$$

This equation, which is that of the straight line portion of the curve in the thick-film region, may be used for estimating the coefficient of friction.

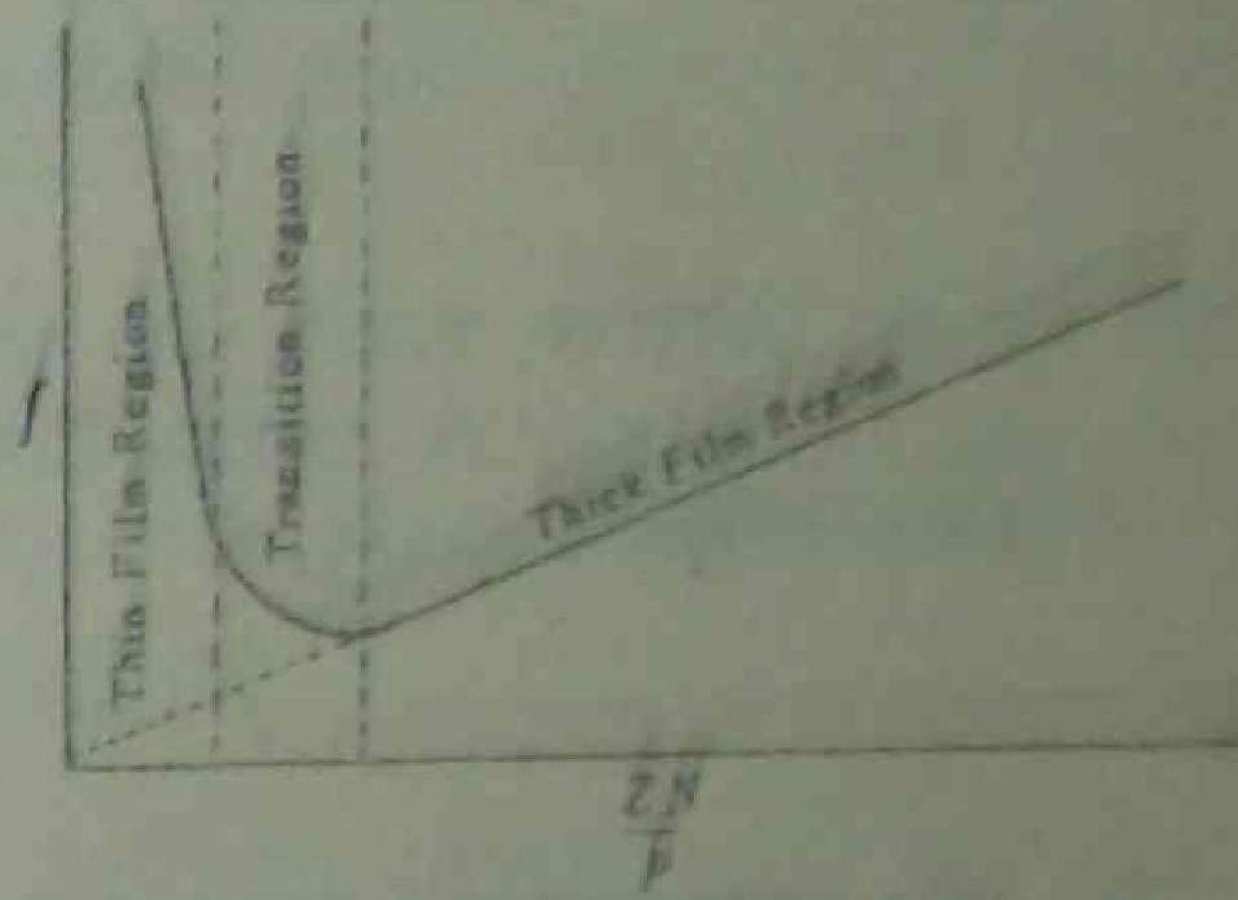


Fig. 23-5

Experimental data indicate that the value of k may be taken as 0.002 for L/D ratios from 0.75 to 2.8. Fig.23-6 shows how k varies, in general, with the L/D ratio. A practical average value for D/C is 1000, and practical values of L/D range from 1 to 2 when space requirements permit using a long bearing. Practical operating values of ZN/p have been determined for a number of typical applications as listed below. The operating value of ZN/p must be sufficiently large to avoid entering the transition or thin-film regions.

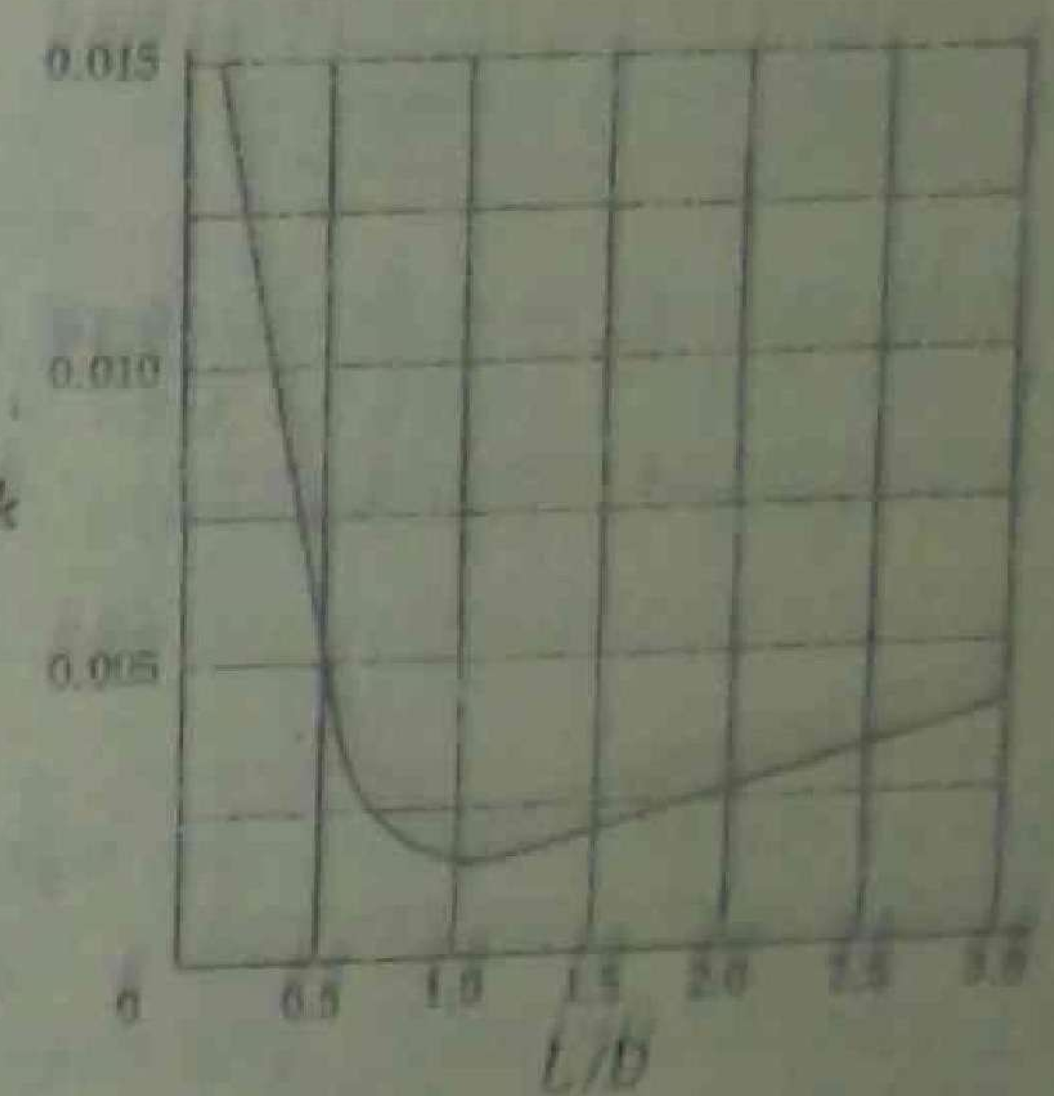


Fig. 23-6

TYPICAL JOURNAL BEARING PRACTICE

Equipment	Bearing	Max Pressure p	Lubricant Z	Lubricant ZN/p
Automobile and Aircraft engines	Main	700-1700	7	15
	Crankpin	1400-3400	8	10
	Wrist pin	2000-5000	8	
Gas and Oil Engines	Main	500-1200	20	20
	Crankpin	1000-1800	40	10
	Wrist pin	1200-2000	65	
Marine Engines	Main	500	30	20
	Crankpin	600	40	15
	Wrist pin	1500	50	
Stationary Steam Engines	Main	200-400	15-60	20
	Crankpin	600-1500	30-80	6
	Wrist pin	1800	25-60	
Reciprocating pumps and compressors	Main	250	30	30
	Crankpin	600	50	20
	Wrist pin	1000	80	
Steam turbines	Main	100-300	2-16	100
Rotary motors and pumps	Shaft	100-200	25	200

OTHER METHODS for determining the coefficient of journal friction are based on Petroff's equation or on hydrodynamic theory.

PETROFF'S EQUATION, which was developed back in 1883, gives an expression for the coefficient of journal friction based on a journal concentric with the bearing (no radial load) and neglects end leakage.

$$f = \frac{\pi^2}{30}\left(\frac{\mu N}{p}\right)\frac{D}{C} = 2\pi^2\left(\frac{\mu N'}{p}\right)\frac{D}{C}$$

Since this equation was derived for an unloaded bearing, it is only an approximation for lightly loaded bearings.

THE SOMMERFELD NUMBER, S, is another dimensionless parameter used extensively in lubrication analysis. Based on hydrodynamic theory, it can be shown that the Sommerfeld number is a function of attitude only, as defined below. It may then be plotted against the quantity $f(D/C)$ which is also a function of attitude only, and the coefficient of journal friction may be determined. The Sommerfeld number is

$$S = \frac{\mu N'}{p}\left(\frac{D}{C}\right)^2$$

One of the main factors that Petroff's equation fails to take into account is the eccentricity of the bearing when under load. The Sommerfeld number when plotted against $f(D/C)$ in accordance with hydrodynamic theory takes this eccentricity into account. The center of the journal when under load is not concentric with the bearing, but moves approximately along a semicircular arc of diameter $C/2$. This results in the establishment of a **minimum film thickness**, h_o, as shown in Fig. 23-7 (shown greatly exaggerated). The distance between the bearing center and the shaft center is called the eccentricity and is denoted by e. The ratio of this eccentricity to the radial clearance is called the attitude or eccentricity ratio.

$$\text{Attitude } \epsilon = \frac{2e}{C} = 1 - \frac{2h_o}{C}$$

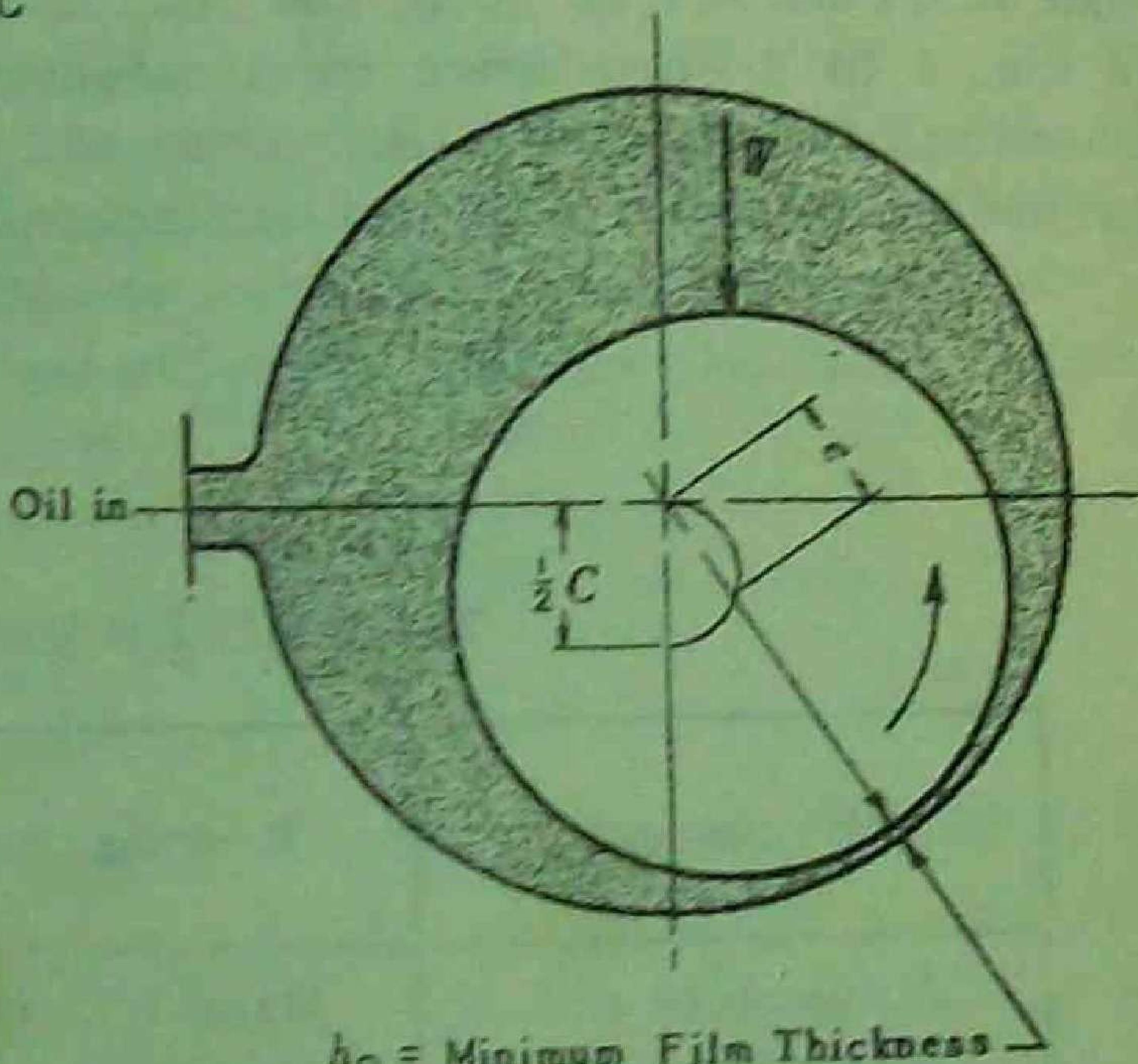


Fig. 23-7

It should be noted that both the Petroff equation and early plots of $f(D/C)$ versus the Sommerfeld number were based on ideal bearings (no end leakage). Several design methods have proposed the use of end leakage and eccentricity correction factors in conjunction with these equations. In the past most of these methods have been rather inaccurate and not entirely satisfactory.

[illegible] CURVES corrected for end leakage and for various L/D ratios in which performance variables are plotted versus the Sommerfeld number have been prepared by A. A. Raimondi and John Boyd of the Westinghouse Research Laboratories. (ASLE Transactions Volume 1, No. 1, April 1958). Their latest curves were developed on a completly rational basis and the results were obtained by the use of computers. These curves supersede previous curves published by the same authors in 1951. Their previous curves required correction factors for end leakage. The use of these latest curves eliminates completely the need for applying end leakage factors and thus greatly simplifies the task of calculating bearing performance. A portion of these charts arbitrarily chosen for a full journal bearing (360°) having an L/D ratio of one have been approximately reproduced to demonstrate their use. These curves are also based on the possibility that rupture of the film may occur.

The reader should refer to the original article for partial bearings and [illegible] design of a bearing for optimum performance. The original article also covers the case of submerged bearings operating under pressure where film rupture is not likely to occur. Note use of a linear plot in the corner block of the following curves.

COEFFICIENT OF FRICTION may be determined from Fig. 23-8 below, where the coefficient of friction variable $f(D/C)$ is plotted against the Sommerfeld number.

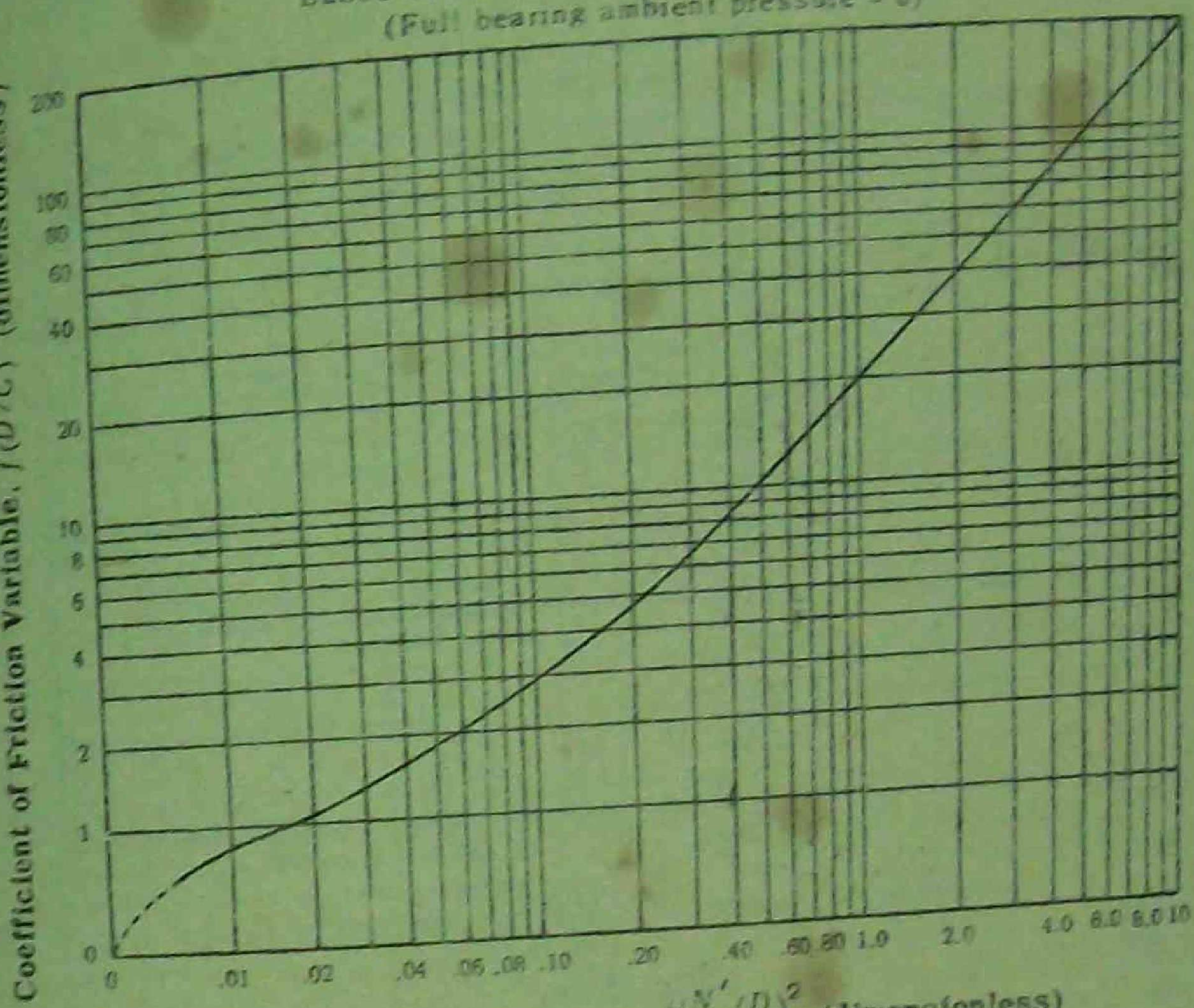


Fig. 23-8

MINIMUM FILM THICKNESS may be determined from Fig. 23-9 below, where the minimum film thickness variable $2h_o/C$ is plotted against the Sommerfeld number.

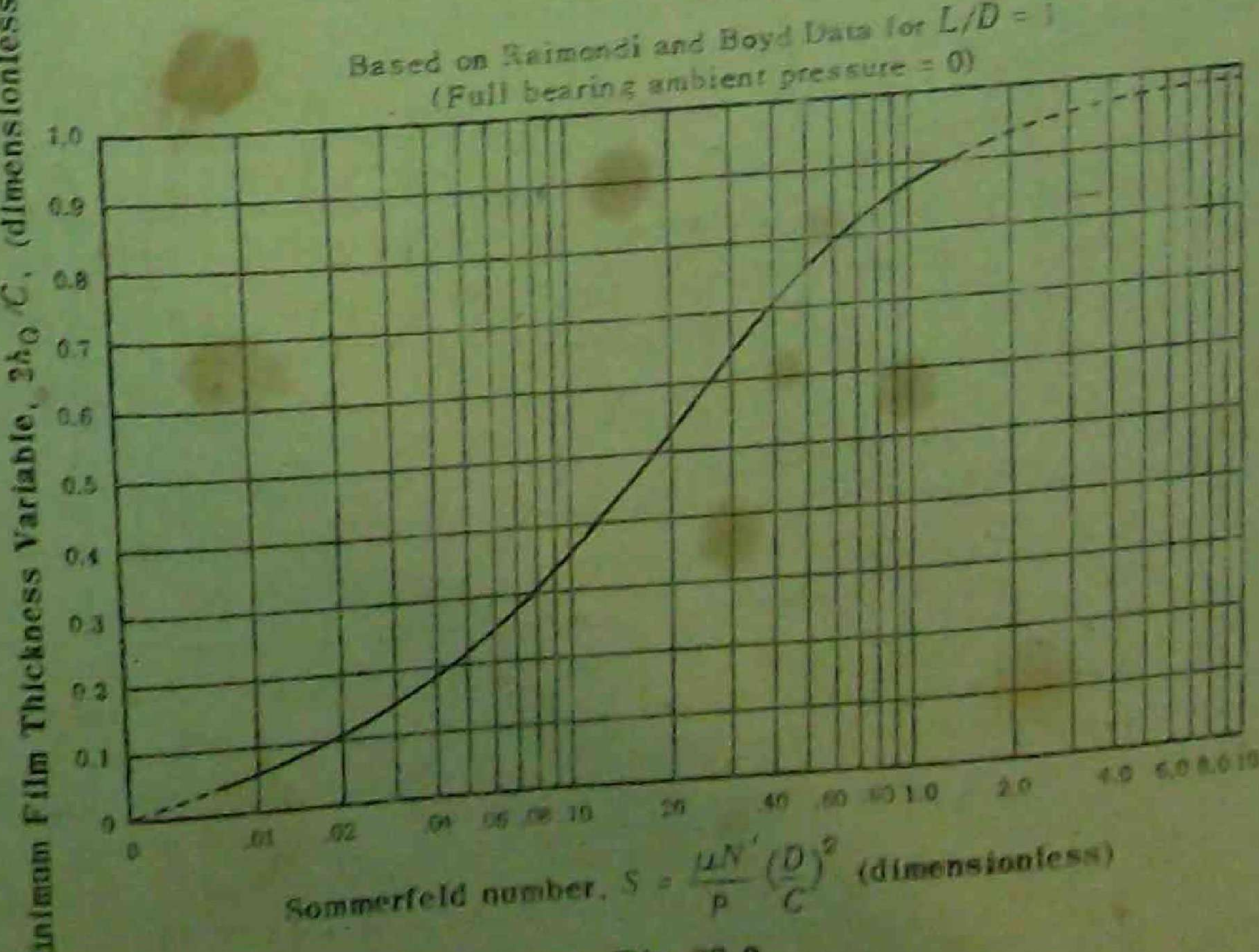


Fig. 23-9

$s_s = \dfrac{VQ}{Ib}$ where

V = transverse shear load on the cross section, lb

b = width of the section containing the critical point, in.

Q = moment of the cross-sectional area of the member, above or below the critical point, with respect to the neutral axis, in^3.

$s_s(\text{max}) = \dfrac{4V}{3A}$ for a circular cross section, and occurs at the neutral axis.

$s_s(\text{max}) = \dfrac{3V}{2A}$ for a rectangular cross section, and occurs at the neutral axis.

$s_n(\text{max})$ = the maximum algebraic stress, psi.

$s_n(\text{min})$ = the minimum algebraic stress, psi.

$\tau(\text{max})$ = the maximum shear stress, psi.

SOLVED PROBLEMS

1. A hypothetical machine member 2" diameter by 10" long and supported at one end as a cantilever will be used to demonstrate how numerical tensile, compressive, and shear stresses are determined for various types of uniaxial loading. In this example note that $s_y = 0$ for all arrangements, at the critical points.

(a) **Axial load only.**

In this case all points in the member are subjected to the same stress.

$A = \pi$ in^2

$s_x = +\dfrac{P}{A} = +\dfrac{3000}{\pi} = +954$ psi

$\tau_{xy} = 0$

$s_n(\text{max}) = s_x = +954$ psi (tension)

$\tau(\text{max}) = \frac{1}{2}(954) = 477$ psi (shear)

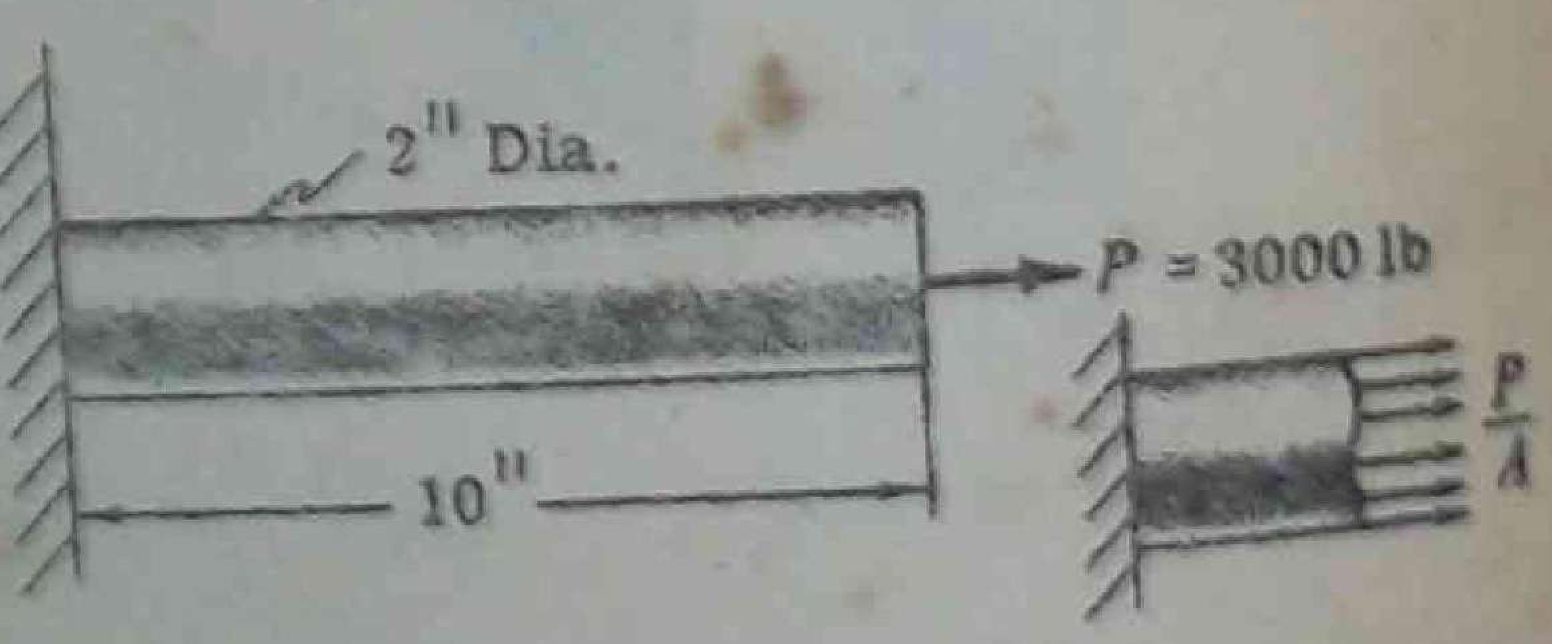


Fig. 2-4

(b) **Bending only.**

Points A and B are critical.

$\tau_{xy} = 0$ at points A and B (no transverse shear).

$s_x = +\dfrac{Mc}{I} = +\dfrac{(600)(10)(1)(64)}{\pi 2^4} = +7650$ psi at point A

$s_x = -\dfrac{Mc}{I} = -7650$ psi at point B

$s_n(\text{max}) = +7650$ psi (tension at point A)

$s_n(\text{min}) = 0$ at point A

$s_n(\text{max}) = 0$ at point B

$s_n(\text{min}) = -7650$ psi (compression at point B)

$\tau(\text{max}) = \frac{1}{2}(7650)$

$= 3825$ psi (shear at points A and B)

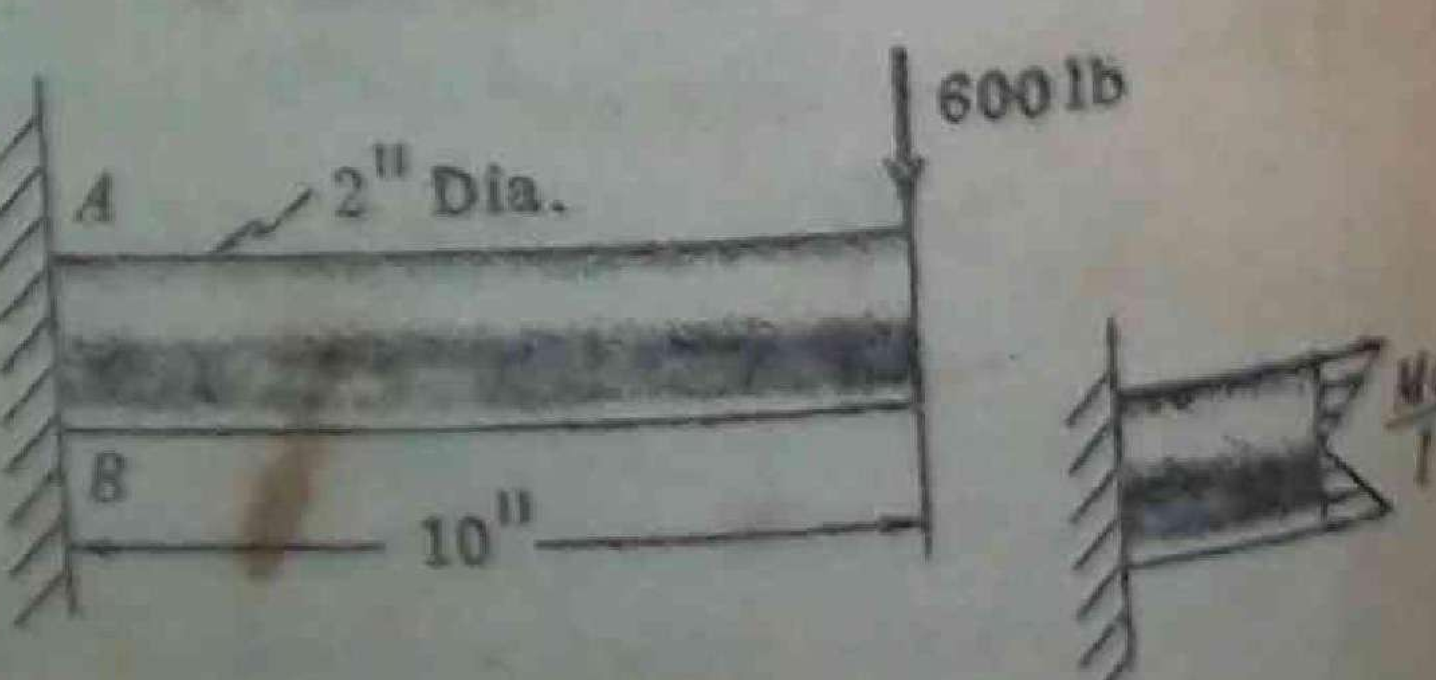


Fig. 2-5

Belt Drives

FLAT BELTS AND V-BELTS may be employed to transmit power from one shaft to another where it is not necessary to maintain an exact speed ratio between the two shafts. Power losses due to slip and creep amount to from 3 to 5 percent for most belt drives. In the following discussion it will be assumed that the shafts are parallel. However, both flat and V-belts may be used between non-parallel shafts to meet special requirements. In this case, in order for the belt to stay on the pulleys, it must approach each pulley in a central plane perpendicular to the pulley's axis of rotation.

BELT DESIGN INVOLVES either the proper belt selection to transmit a required power or the determination of the power that may be transmitted by a given flat belt or by one V-belt. In the first case the width of the belt is unknown, while in the second case the width is known. The belt thickness is assumed for both cases.

The power transmitted by a belt drive is a function of the belt tensions and belt speed.

$$\text{Power} = \frac{(T_1 - T_2)v}{550} \quad \text{in horsepower}$$

where T_1 = belt tension in tight side, lb
T_2 = belt tension in loose side, lb
v = belt speed, ft/sec.

The following formula for determining the stress, s_2 psi, for flat belts applies when the thickness of the belt is given but the width is unknown.

$$\frac{s_1 - w'v^2/g}{s_2 - w'v^2/g} = e^{f\alpha}$$

where s_1 = maximum allowable stress, psi
s_2 = stress in slack side of belt, psi
w' = weight of 1 ft of belt 1 in^2 in cross section
v = belt velocity, ft/sec
g = acceleration due to gravity, 32.2 ft/sec^2
f = coefficient of friction between belt and pulley
α = angle of wrap of belt on pulley, radians.

The required cross section area of the belt for the case of the width unknown may be determined by

$$\frac{T_1 - T_2}{s_1 - s_2} = \text{required cross section area}$$

The required belt width b is therefore b = area/thickness. The value of $(T_1 - T_2)$ may be determined from the horsepower requirement, hp = $(T_1 - T_2)v/550$.

The maximum tension in the tight side of the belt depends on the allowable stress of the belt material. Leather and cotton duck impregnated with rubber built up in plies are generally used. The allowable tensile stress for leather belting is usually 300 to 500 psi, and the allowable stress for rubber belting will run from 150 to 250 psi, depending on the quality of the material. Leather belting can be obtained in single ply thickness of 1/8 in., 5/32 in., and 3/16 in. Double and triple ply belts are also available in multiples of these increments. The specific weight of leather is about 0.035 lb/in^3. Rubber belting can be obtained in ply thicknesses of 3/64 to 5/64 in. and has a specific weight of about 0.045 lb/in^3.

The following formula for determining the value of T_2 for both flat and V-belts applies when the width and thickness of the belt are known.

$$\frac{T_1 - wv^2/g}{T_2 - wv^2/g} = e^{f\alpha/\sin\frac{1}{2}\theta}$$

where w = the weight of 1 ft of belt; v = belt velocity, ft/sec; g = acceleration due to gravity, 32.2 ft/sec^2; f = coefficient of friction between belt and pulley; α = angle of wrap, radians; θ = groove angle for the V-belt (θ is 180° for a flat belt).

The quantity wv^2/g is due to centrifugal force, which tends to cause the belt to leave the pulley and reduces the power that may be transmitted.

THE LOAD CARRYING CAPACITY of a pair of pulleys is determined by the one which has the smaller $e^{f\alpha/\sin\frac{1}{2}\theta}$. It is for this reason that a V-belt may be used with one grooved pulley and one flat pulley, saving the expense of unnecessary machining.

Excessive flexing of a belt will result in a shortened life. A minimum ratio of the diameter of a pulley to the thickness of the belt is about 30 for reasonable life.

SELECTION OF BELTS can be made on the basis of application of the appropriate equations or by use of tables supplied by the American Leather Belting Association for leather belts and by use of catalogs supplied by the various V-belt manufacturers. In this book, the application of the equations will be used, although the recommendations of the A.L.B.A. or V-belt manufacturers will generally give safer designs incorporating suitable application factors.

ANGLES OF WRAP: The angles of wrap for an open belt may be determined by:

$$\sin\beta = \frac{R - r}{C}$$

$$\alpha_1 = 180° - 2\beta = 180° - 2\sin^{-1}\frac{R-r}{C}, \qquad \alpha_2 = 180° + 2\beta = 180° + 2\sin^{-1}\frac{R-r}{C}$$

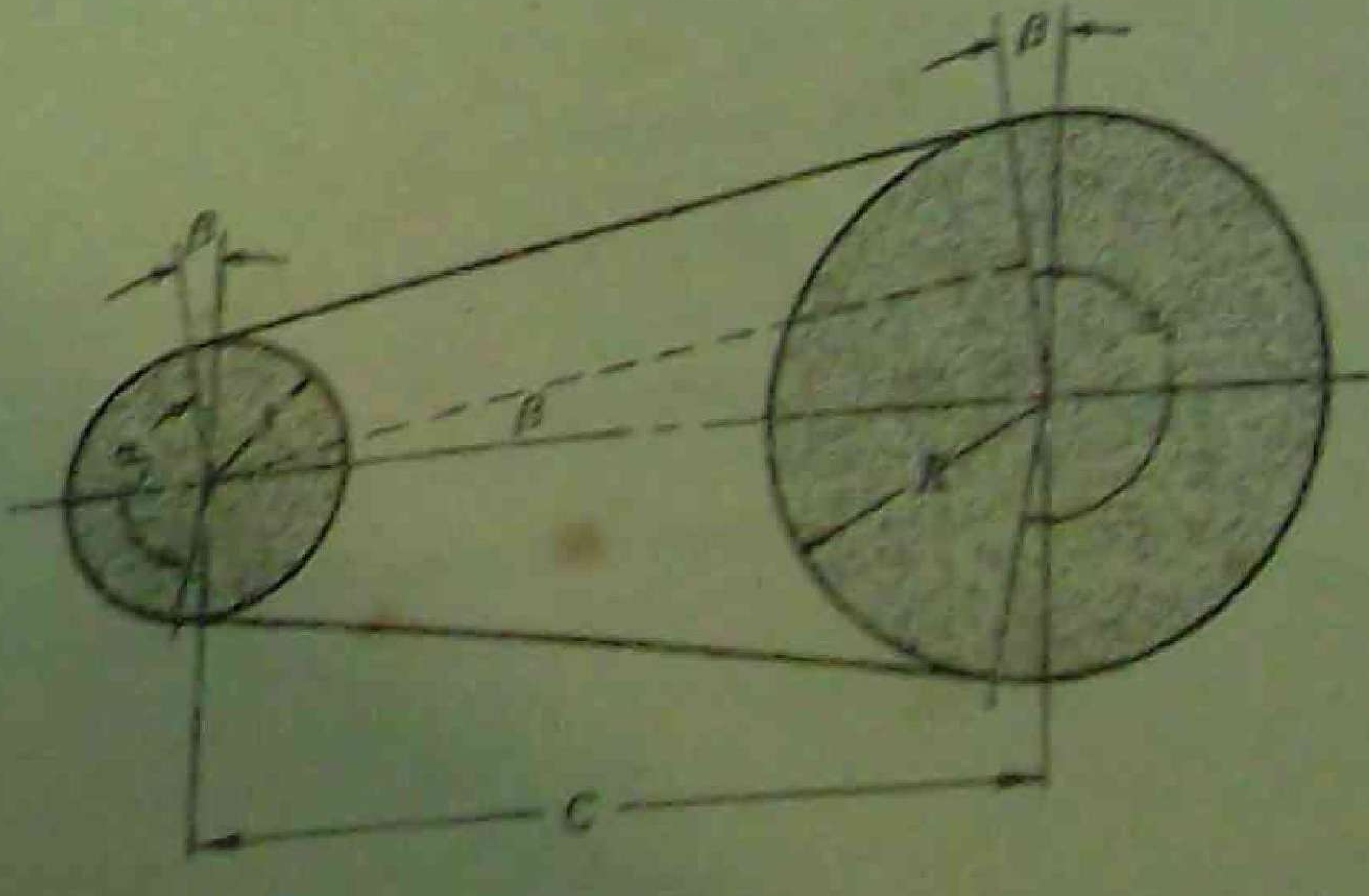


Fig. 24-1

The angles of wrap for a crossed belt drive may be determined by:

$$\sin \beta = \frac{R+r}{C}$$

$$\alpha_1 = \alpha_2 = 180^\circ + 2\beta$$
$$= 180^\circ + 2\sin^{-1}\frac{R+r}{C}$$

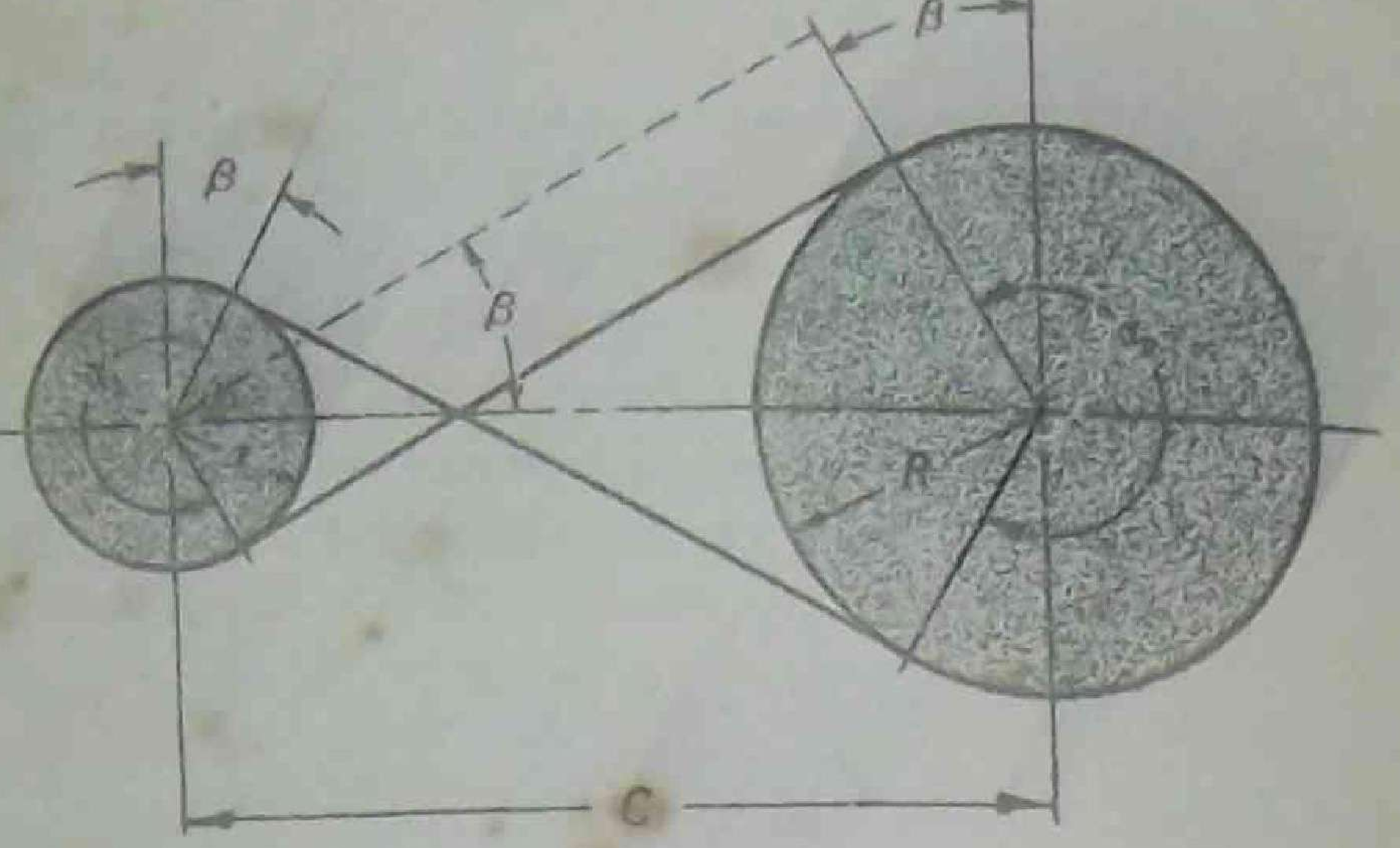


Fig. 24-2

SOLVED PROBLEMS

1. Derive $\dfrac{T_1 - wv^2/g}{T_2 - wv^2/g} = e^{f\alpha}$ for a flat belt.

Solution:

(a) Consider a differential element of belt. The forces acting on the differential element are (1) tensions T and $(T+dT)$, (2) centrifugal force $(wv^2/g)\,d\phi$, (3) normal force dN, (4) frictional force $f\,dN$.

Note that there is no bending moment and no shear force acting on the belt. The belt is a flexible member and cannot sustain shear and bending of any appreciable magnitude compared to the other forces.

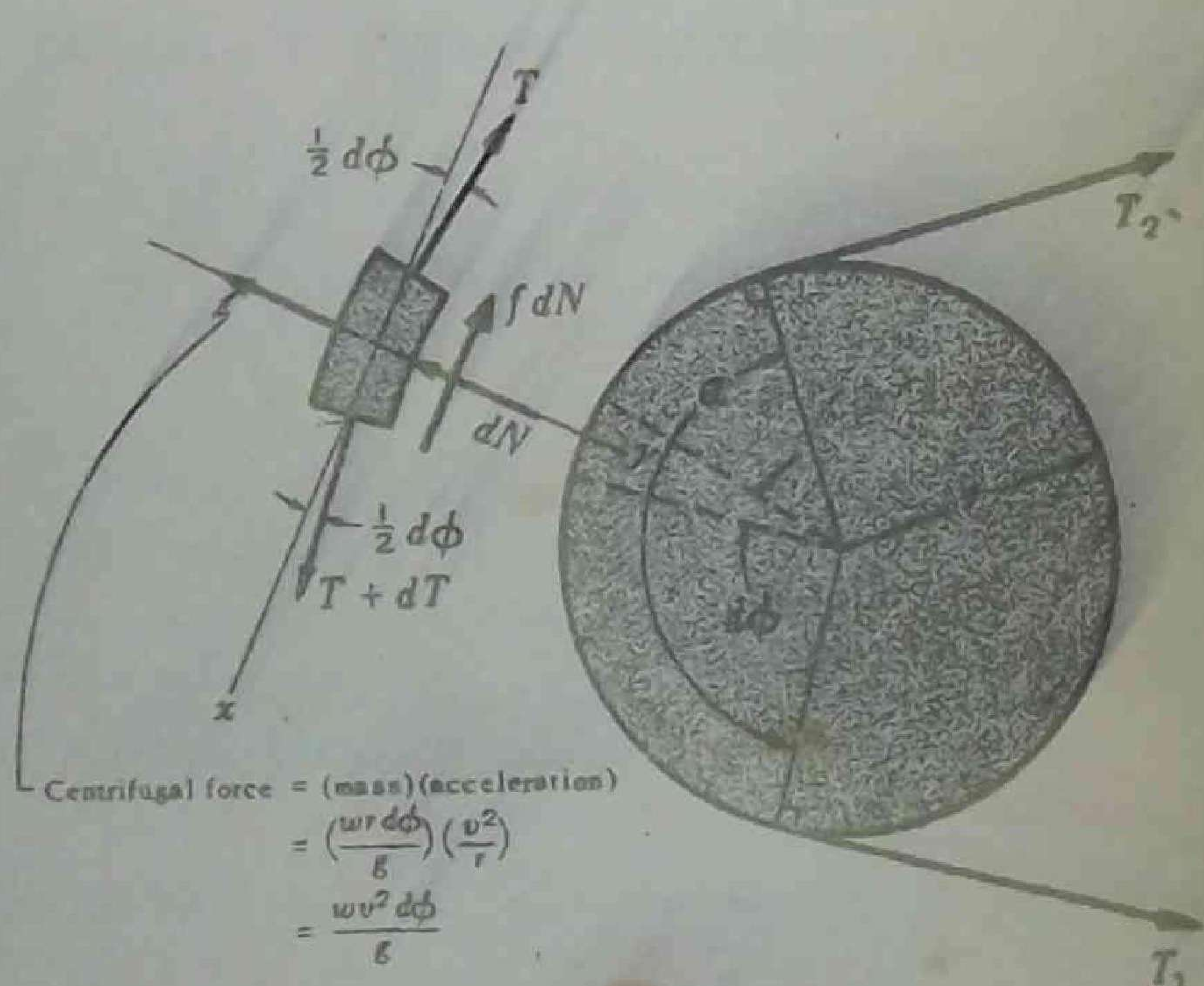


Fig. 24-3

(b) Setting up an arbitrary x and y direction for the element of belt and considering the element in equilibrium since the inertia force is included,

$$\Sigma F_x = 0 \quad \text{or} \quad (T+dT)\cos\tfrac{1}{2}d\phi - f\,dN - T\cos\tfrac{1}{2}d\phi = 0$$
$$\Sigma F_y = 0 \quad \text{or} \quad (T+dT)\sin\tfrac{1}{2}d\phi + T\sin\tfrac{1}{2}d\phi - (wv^2/g)\,d\phi - dN = 0$$

(c) Since $\cos\frac{1}{2}d\phi = 1$ and $\sin\frac{1}{2}d\phi = \frac{1}{2}d\phi$ in the limit,

1) $(T+dT)(1) - f\,dN - T = 0$ or $dN = dT/f$

2) $(T+dT)(\frac{1}{2}d\phi) + T(\frac{1}{2}d\phi) - dN - (wv^2/g)\,d\phi = 0$

Substituting $dN = dT/f$ from (1) into (2), and dropping out differentials of the second order,

$$T\,d\phi - dT/f - (wv^2/g)\,d\phi = 0$$

(d) Then $\dfrac{dT}{T - wv^2/g} = f\,d\phi$, $\displaystyle\int_{T_2}^{T_1} \frac{dT}{T - wv^2/g} = \int_0^{\alpha} f\,d\phi$, and finally $\dfrac{T_1 - wv^2/g}{T_2 - wv^2/g} = e^{f\alpha}$.

(See Page 261 for proper units.)

4. A fan is driven by a belt from a motor which runs at 880 rpm. A medium double ply leather belt 5/16 in. thick and 10 in. wide is used. The diameters of the motor pulley and driven pulley are respectively 14 in. and 54 in. The center distance is 54 in., and both pulleys are made of cast iron. Coefficient of friction of leather on cast iron is 0.35. The allowable stress for the belt is 350 psi, which allows for the factor of safety and also for the fact that a double ply belt does not have double the capacity of a single ply belt. (A double ply belt has approximately 85% the capacity of a single ply belt of the same thickness.) The belt weighs 0.035 lb/in^3. What is the horsepower capacity of the belt?

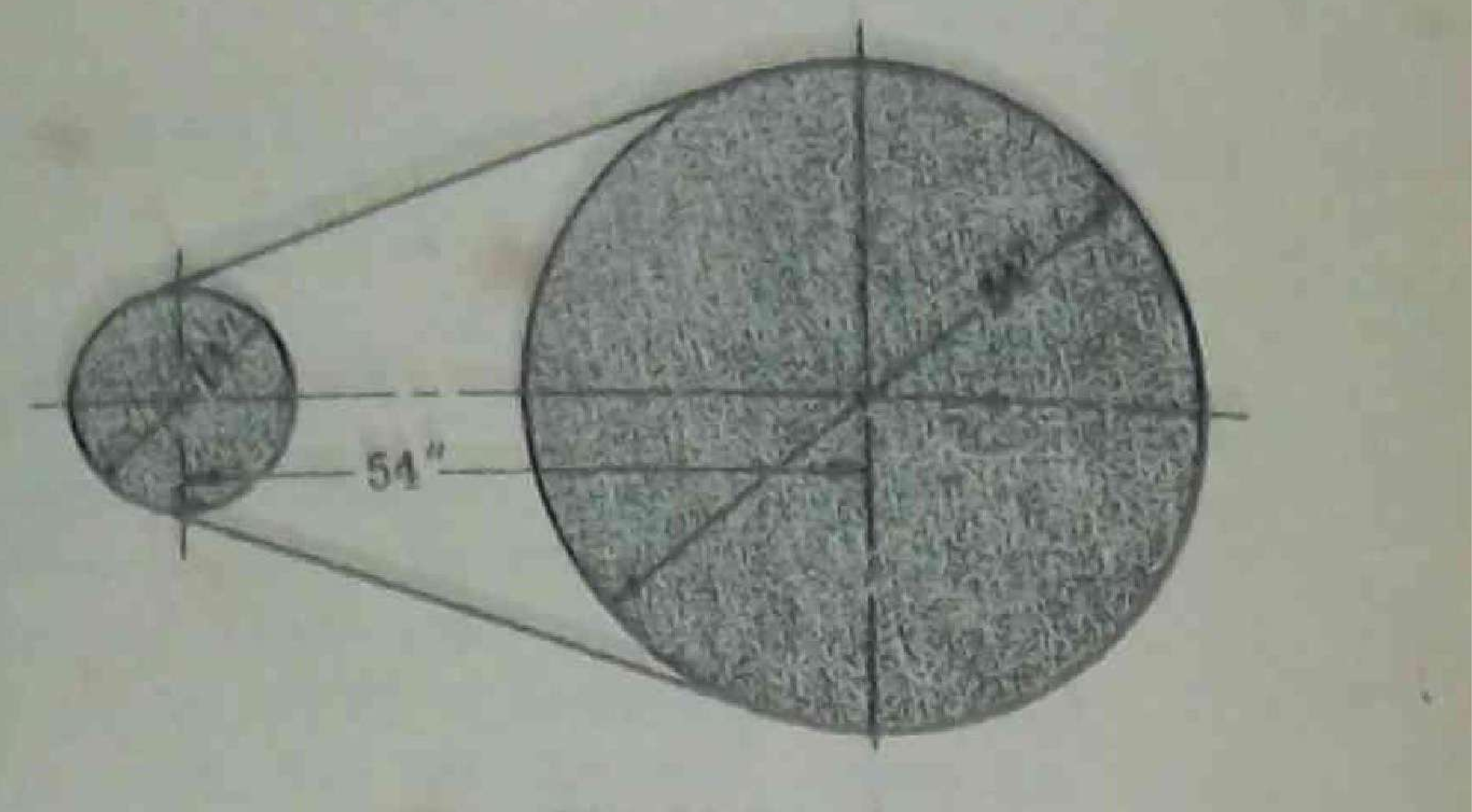


Fig. 24-6

Solution:

Angles of wrap of smaller and larger pulleys are respectively

$$\alpha_1 = 180^\circ - 2\sin^{-1}(R-r)/C = 180^\circ - 2\sin^{-1}(27-7)/54 = 136.6^\circ$$

$$\alpha_2 = 180^\circ + 2\sin^{-1}(R-r)/C = 223.4^\circ$$

The pulley which governs the design is the one with the smaller $e^{f\alpha}$. Here the smaller pulley governs, i.e. the smaller pulley is transmitting its maximum power with the belt on the point of slip while the larger pulley is not developing its maximum capacity. Then

$$\frac{T_1 - wv^2/g}{T_2 - wv^2/g} = e^{f\alpha_1} \qquad \frac{1094 - 117}{T_2 - 117} = e^{0.35\,(136.6\pi/180)} = 2.30, \qquad T_2 = 542 \text{ lb}$$

where $w = 12(10)(5/16)(0.035) = 1.31$ lb/ft, $\quad g = 32.2$ ft/sec^2,

$v = \pi DN = \pi(14/12 \text{ ft})(880/60 \text{ rps}) = 53.7$ ft/sec, $\quad T_1 = (350 \text{ lb/in}^2)(\frac{5}{16} \times 10 \text{ in}^2) = 1094$ lb.

$$\text{Horsepower capacity} = \frac{(T_1 - T_2)v}{550} = \frac{(1094 - 542)(53.7)}{550} = 53.9 \text{ hp}.$$

5. A compressor is driven by a 900 rpm motor by means of a $\frac{3}{8}$" by 10" flat belt. The motor pulley is 12" in diameter and the compressor pulley is 60" in diameter. The shaft center distance is 60" and an idler is used to make the angle of wrap on the smaller pulley 220° and on the larger pulley 270°. The coefficient of friction between the belt and the small pulley is 0.3, and between the belt and the large pulley is 0.25. The maximum allowable belt stress is 300 psi and the belt weighs 0.035 lb/in^3. (*a*) What is the horsepower capacity of this drive? (*b*) Would changing the small pulley to a multiple V-pulley (groove angle $\theta = 34^\circ$ and coefficient of friction 0.25) using the same compressor pulley, and eliminating the idler pulley, provide a more effective drive with greater horsepower capacity? Assume that the pitch diameter of the V-belt and the pitch diameter of the large pulley remain the same as for the flat belt arrangement: 12" and 60". Assume also that the total of the maximum force in each belt is the same as for the flat belt (i.e. T is constant) and that the centrifugal effect of all the belts is the same as for the flat belt.

Solution:

(*a*) For small pulley, $e^{f\alpha} = e^{0.3(220\pi/180)} = 3.16$; for large pulley, $e^{f\alpha} = e^{0.25(270\pi/180)} = 3.26$. Hence the small pulley governs. Then

$$\frac{T_1 - wv^2/g}{T_2 - wv^2/g} = e^{f\alpha}, \qquad \frac{1125 - 108}{T_2 - 108} = 3.16, \qquad T_2 = 430 \text{ lb}$$

where $T_1 = (300)(10)(3/8) = 1125$ lb, $\quad w = 12(10)(3/8)(0.035) = 1.57$ lb/ft, $\quad v = \pi(1)(15) = 47.1$ fps.

$$\text{Horsepower capacity} = \frac{(T_1 - T_2)v}{550} = \frac{(1125 - 430)(47.1)}{550} = 59.6 \text{ hp}$$

(b) For an open belt arrangement with no idler: angle of wrap on smaller pulley = 180° − 2 sin⁻¹ [illegible] = 132.8°, on larger pulley = 227.2°. Now

for small pulley (V-belt in the groove), $e^{f\alpha/\sin\frac{1}{2}\theta}$ = [illegible]

for large pulley (V-belt on a flat pulley), $e^{f\alpha} = e^{0.25(227.2\pi/180)} = 2.69$

Thus, although the capacity of the small pulley is increased, the larger pulley is now the limiting with $e^{f\alpha} = 2.69$. Using $\frac{1125-108}{T_2-108} = 2.69$ or $T_2 = 486$ lb, the new (decreased) horsepower capacity is $\frac{(1125-486)(47.1)}{550} = 54.8$ hp.

6. A crossed belt drive is to transmit 10 hp at 1000 rpm of the smaller pulley. The smaller pulley has diameter 10", the velocity ratio is 2, and the center distance is 50". It is desired to use a flat belt $\frac{1}{4}$" thick with an expected coefficient of friction 0.3. If the maximum allowable stress in the belt is 250 psi, determine the necessary leather belt width b. The leather weighs 0.035 lb/in³.

Fig. 34.7

Solution:

Both pulleys have the same angle of wrap α and the same horsepower capacity.

$$\alpha = 180 + 2\sin^{-1}(R+r)/C = 180 + 2\sin^{-1}(10+5)/50 = 214.8°$$

$$\frac{s_1 - w'v^2/g}{s_2 - w'v^2/g} = e^{f\alpha}, \qquad \frac{250-24.8}{s_2-24.8} = e^{[illegible]}, \qquad s_2 = 97.9 \text{ psi}$$

where $w' = 12(0.035) = 0.42$ lb/ft-in², $v = \pi(10/12)(1000/60) = 43.6$ ft/sec

Using $(T_1 - T_2) = \frac{(hp)(550)}{v} = \frac{(10)(550)}{43.6} = 126$ lb, $A = \frac{T_1 - T_2}{s_1 - s_2} = \frac{126}{250 - 97.9} = 0.83$ in² and the belt width $b = (0.83 \text{ in}^2)/(\frac{1}{4} \text{ in.}) = 3.32$ in. Use $3\frac{1}{2}$ in belt width.

7. A V-belt drive is to transmit 35 hp from a 10 inch pitch diameter sheave operating at 1800 rpm to a 36 inch diameter flat pulley. The center distance between the input and output shafts is 40". The groove angle $\theta = 40°$, and the coefficient of friction for the belt and sheave is 0.2, and the coefficient of friction between the belt and flat pulley is 0.2. The cross section of the belt is $b_2 = 1.5"$ wide at the top and $b_1 = 0.75"$ wide at the bottom by $d = 1.0"$ deep. Each belt weighs 0.04 lb/in³ and the allowable tension per belt is 200 lb. How many belts are required? (Note: analyze for one belt first.)

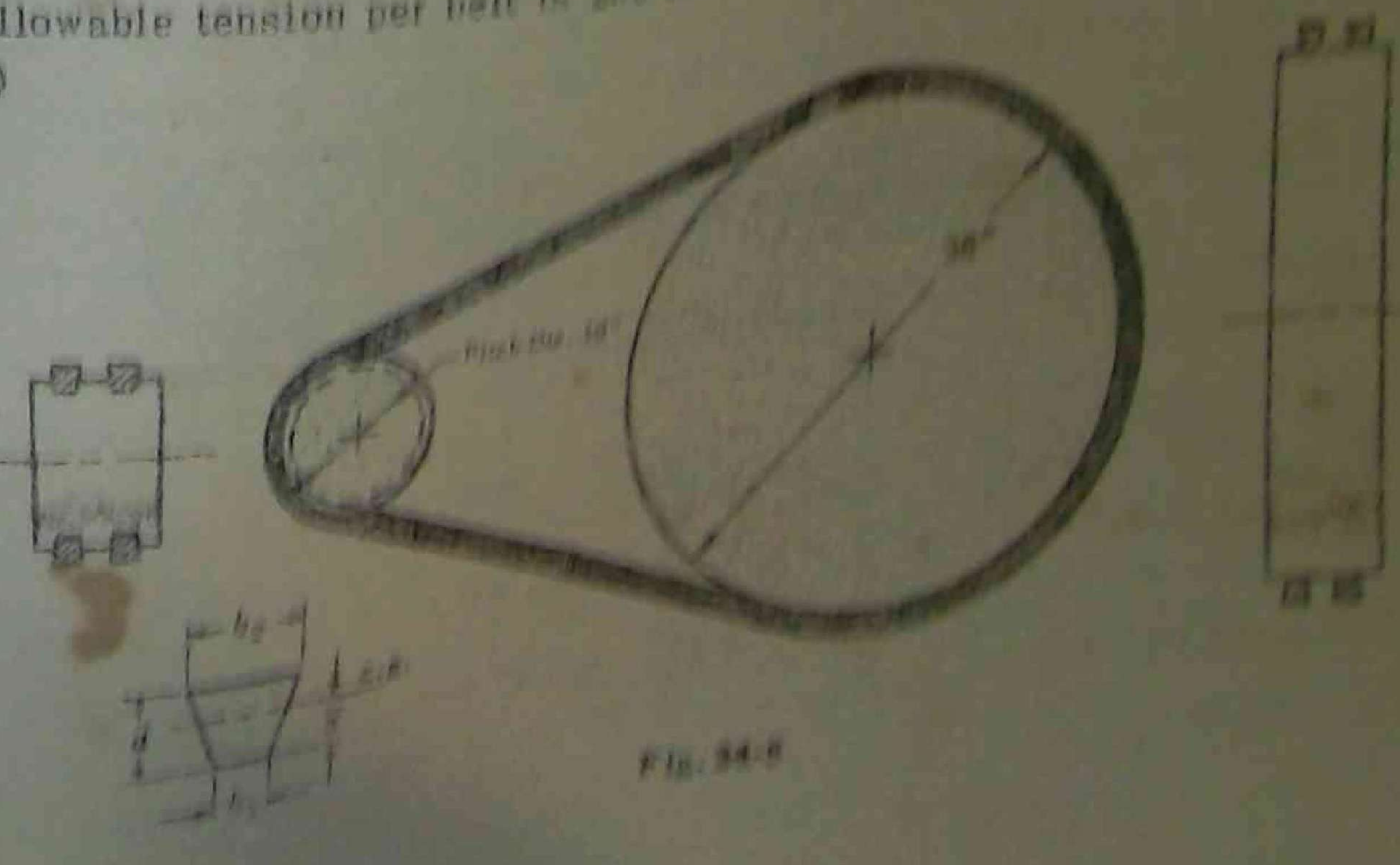


Fig. 34.8

Solution:

While the thickness of a flat belt is ordinarily negligible with respect to the diameter of a pulley, the thickness of a V-belt may not be negligible. Determine the pitch diameter of the V-belt on the flat pulley by assuming that the pitch diameter is measured to the centroid of the belt section.

Distance from base to C.G. is $\bar{x} = \dfrac{d(b_1 + 2b_2)}{3(b_1 + b_2)} = \dfrac{1(0.75 + 2\times 1.5)}{3(0.75 + 1.5)} = 0.56$ in.

Pitch diameter of larger pulley $= 36 + 2(0.56) = 37.1''$, or pitch radius $= 18.6''$.

For small pulley, $\alpha = 180^\circ - 2\sin^{-1}(18.6 - 5)/40 = 140.4^\circ$. For large pulley, $\alpha = 219.6^\circ$.

Compare the capacities, $e^{f\alpha/\sin\frac{1}{2}\theta}$, of the two pulleys:

Small, $e^{0.2(140.4\pi/180)/\sin 20^\circ} = 4.18$. Large, $e^{0.2(219.6\pi/180)/\sin 90^\circ} = 2.15$.

The larger pulley governs the design. (Note that the angle of the V-belt is slightly larger than 40°, but the belt will wedge into the 40° groove.)

The area of one belt $= \frac{1}{2}(b_1 + b_2)(d) = \frac{1}{2}(0.75 + 1.5)(1) = 1.125$ in^2, and the tension on the slack side of a belt is found from

$$\frac{T_1 - wv^2/g}{T_2 - wv^2/g} = e^{f\alpha/\sin\frac{1}{2}\theta} \qquad \frac{200 - 103.5}{T_2 - 103.5} = 2.15, \qquad T_2 = 148 \text{ lb}$$

where $w = 12(1.125)(0.04) = 0.54$ lb/ft, $v = \pi(10/12)(1800/60) = 78.5$ fps.

Horsepower per belt $= (T_1 - T_2)v/550 = (200 - 148)(78.5)/550 = 7.43$ hp/belt.

Number of belts required $= (25 \text{ hp})/(7.43 \text{ hp/belt}) = 3.37$. Use 4 belts.

8. A 9" pulley is keyed to a shaft, and the center plane of the pulley overhangs the nearer bearing by 10", as shown in Fig. 24-9 below. An open belt arrangement is used. The pulley is driven by an 1800 rpm motor through a flat belt with a 1 to 1 velocity ratio of the pulleys. The belt is $\frac{3}{8}$" by 6", and weighs 0.035 lb/in^3. The coefficient of friction between the belt and pulleys is 0.3. The belt is run at its maximum capacity with a maximum belt stress of 300 psi.

It is decided that the horsepower capacity is to be doubled; and, of the several possibilities, this problem will concern itself with the effect of increasing the belt width. Assume that the belt width is to be increased, with all other conditions remaining the same. (*a*) How much should the belt width be increased to double the horsepower capacity? (*b*) Assuming that the center plane of the pulley remains at the same distance from the nearer bearing (that is, 10"), by how many pounds are the bearing forces increased? The distance between bearings is 18". Assume that power is taken from the shaft through a flexible coupling.

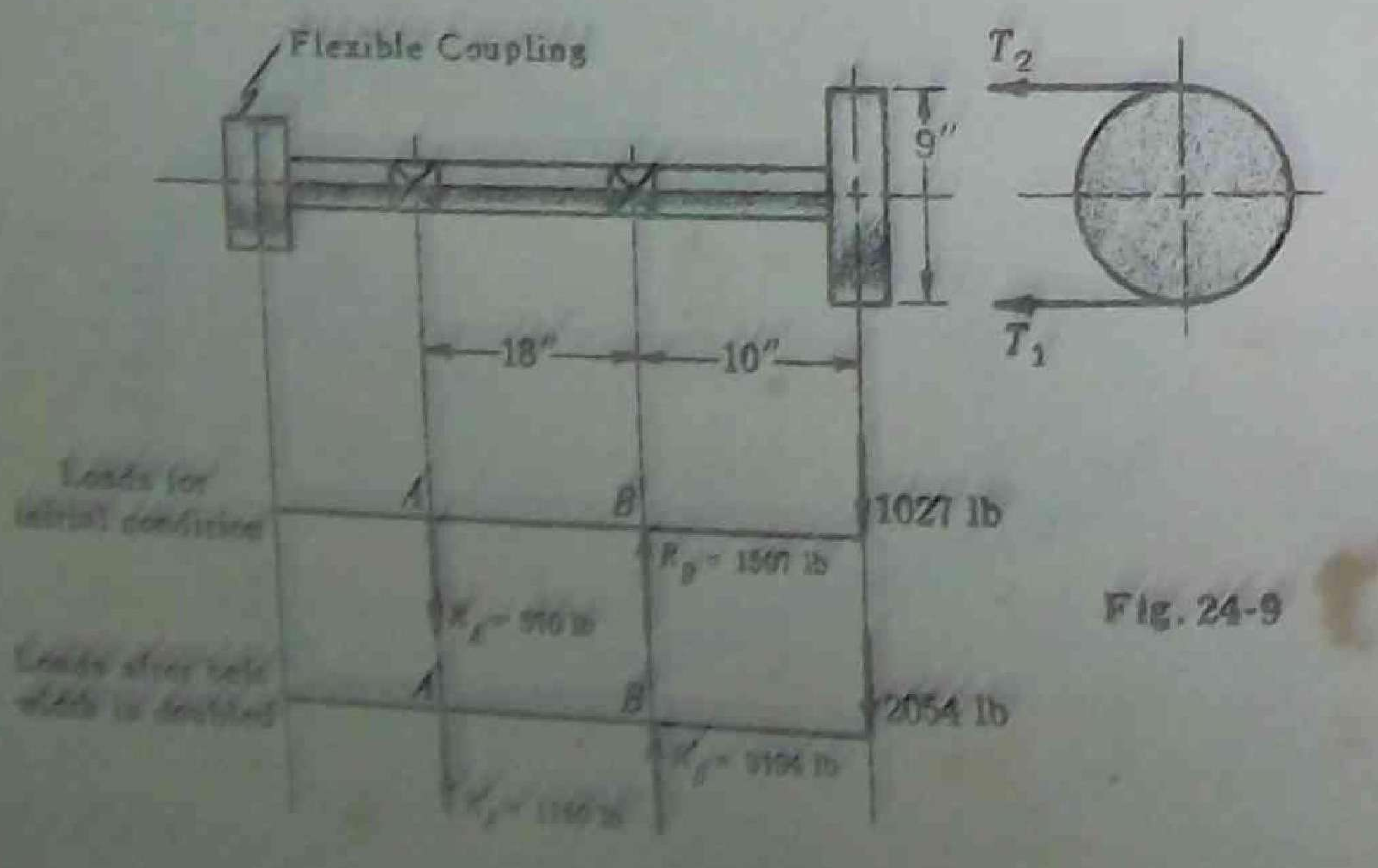


Fig. 24-9

Chapter 25

Welding

INTRODUCTION. There are many phases of welding which are important and rightly deserve a place in machine design considerations. For the designer, the immediate problem is the determination of the size of weld necessary for a given part, and this dictates a stress analysis where parts are subject to load, either static or fluctuating. The procedures as recommended by the American Welding Society (AWS) with modifications as recommended by The Lincoln Electric Company, will be used.

A designer is required to use the design stresses and procedures as specified by the various codes for structures, bridges, and pressure vessels where procedures are conservative. A machine designer, on the other hand, has greater freedom in designing machines, in general. The viewpoint in this chapter is one to permit freedom and flexibility in design.

TYPES OF WELDED JOINTS.

(1) Butt welds. See Fig.25-1.

According to the Lincoln Electric Co., a butt weld, when properly made, has equal or better strength than the plate and there is no need for calculating the stress in the weld or attempting to determine its size. It is necessary to match the electrode strength to plate strength when welding alloy steels.

Several of the codes suggest reducing the strength by some factor, the efficiency of the joint. Where the strength is to be reduced, the equation for the allowable force on a butt weld is given by

$$F_{all} = s_t tLe$$

where

F_{all} = allowable force, lb
s_t = allowable stress for the weld, psi
t = thickness of plate, inches
L = length of the weld, inches
e = efficiency.

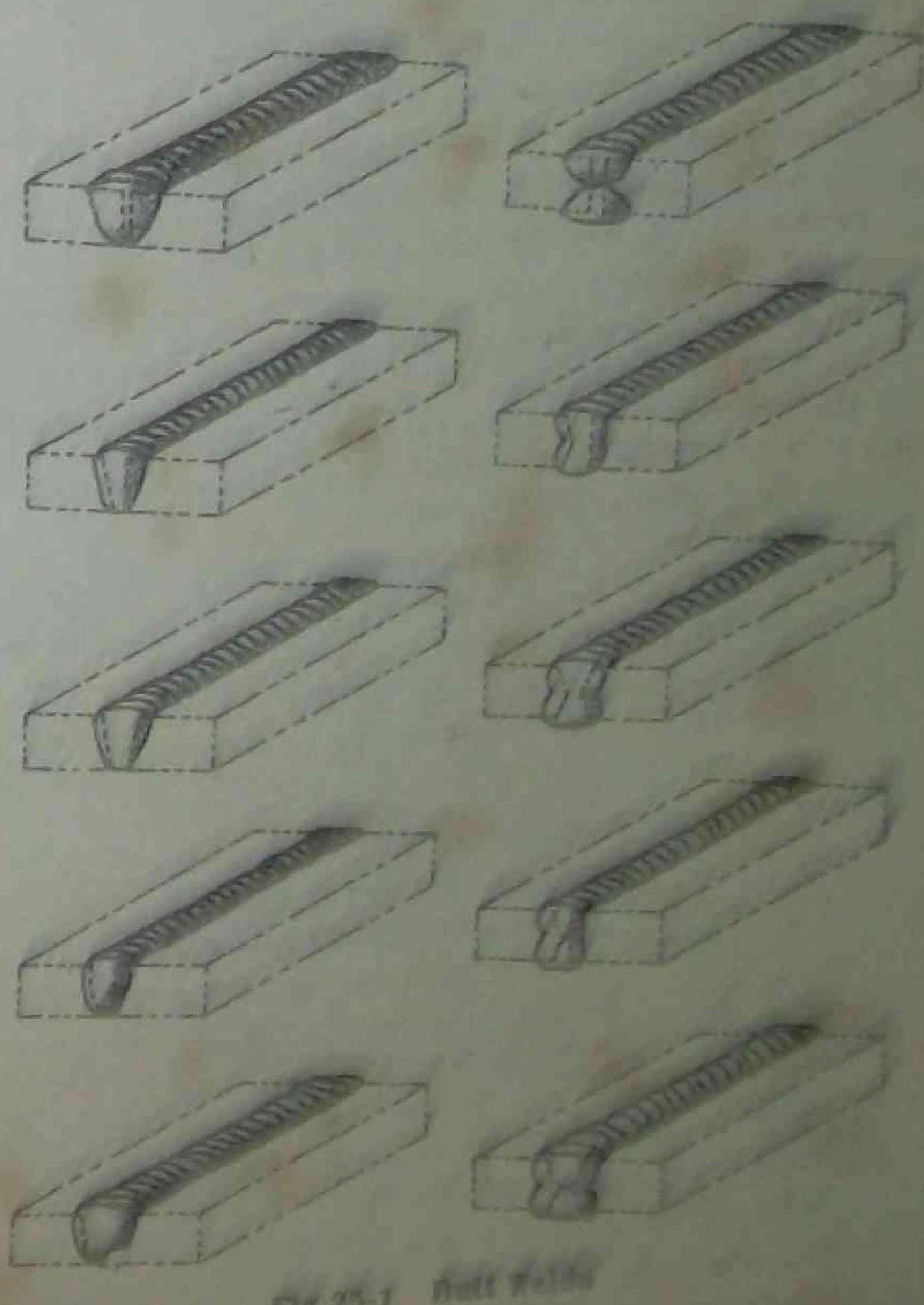


Fig.25-1 Butt Welds

(2) Fillet welds.

Fillet welds are classified according to the direction of the load: (a) parallel load, (b) transverse load. See Fig.25-2 below.

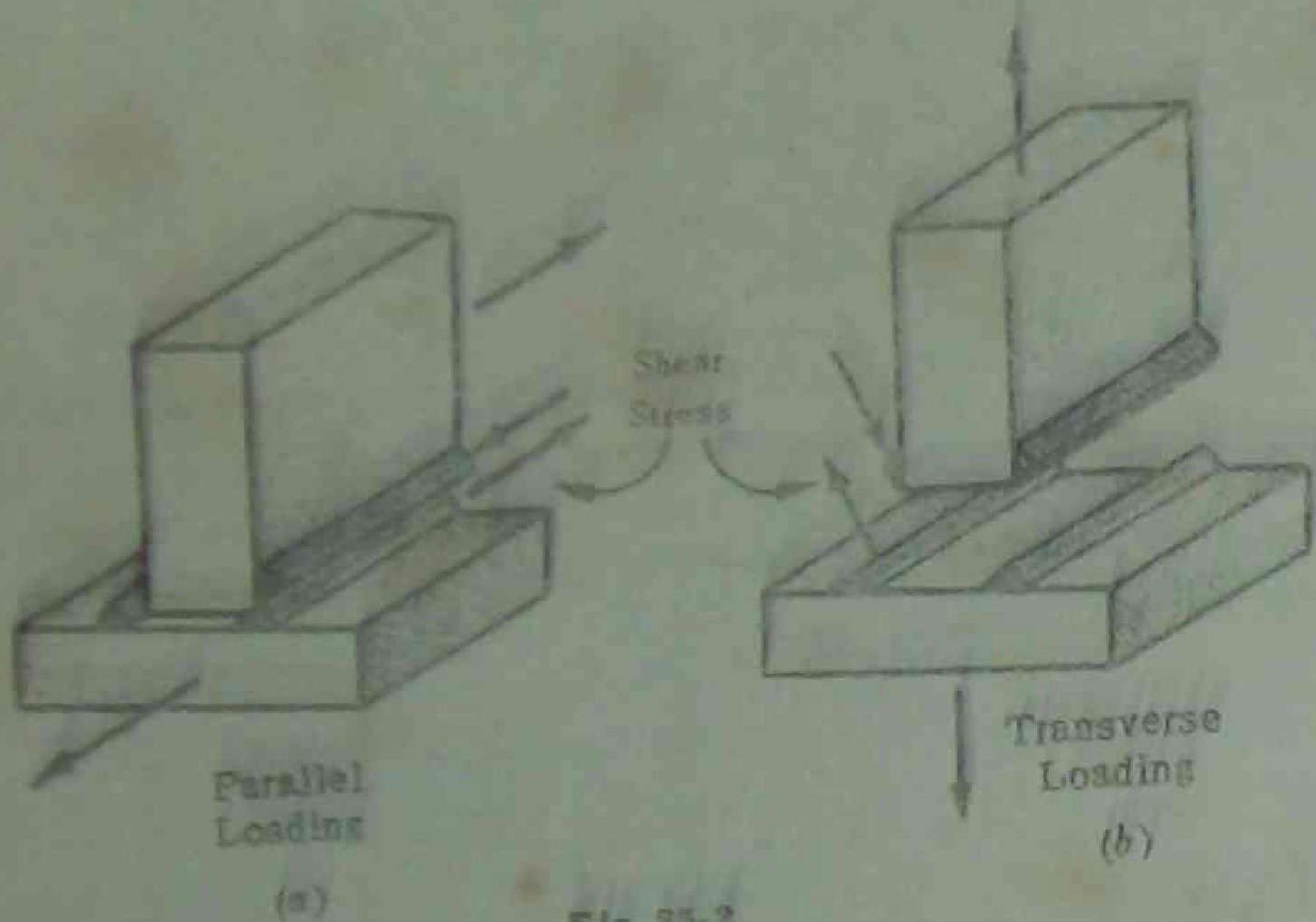


Fig. 25-2

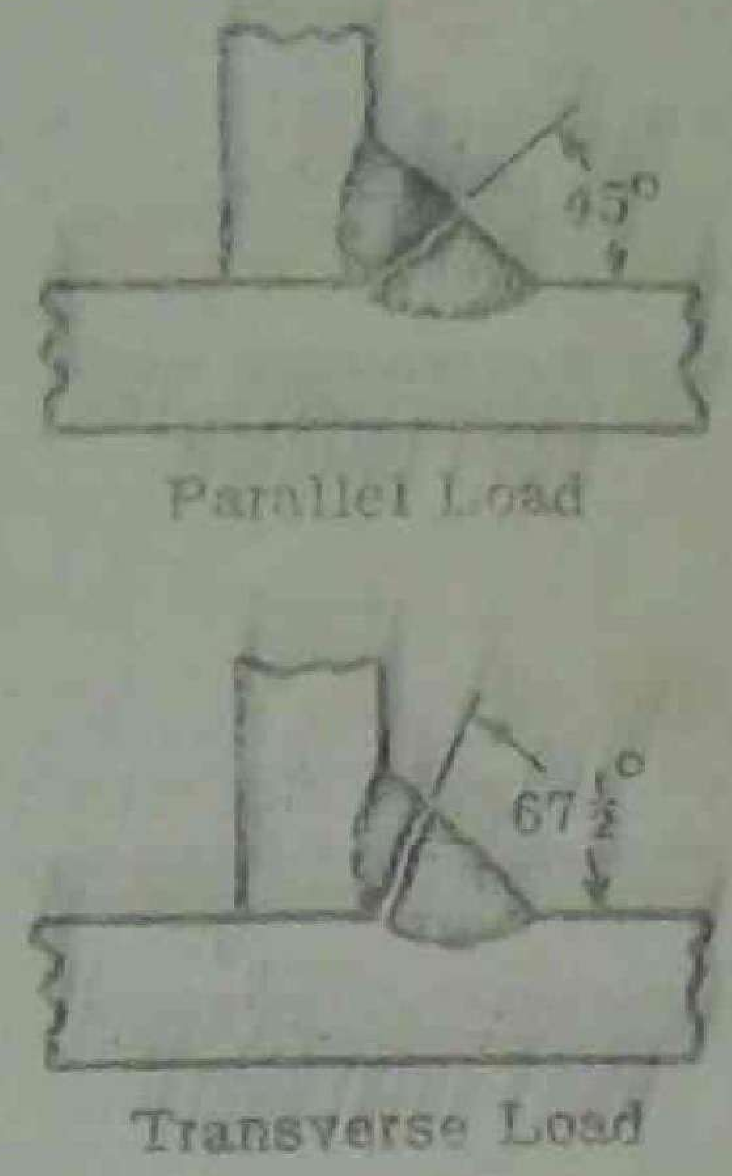


Fig. 25-3

The plane of maximum shear stress in the conventional 45° fillet weld is the 45° throat when subjected to a parallel load and the 67½° throat when subjected to a transverse load, as shown in Fig.25-3 above. This results in greater strength for a transverse load.

Leg size is the basis of specifying a weld in the United States (throat is used in Europe). The size of a fillet weld is specified by the leg length of the largest inscribed isosceles right triangle or the leg lengths of the largest inscribed right triangle.

The leg length of a fillet weld with equal legs is given by w and the leg lengths of a fillet weld with unequal legs are given by a and b, as shown in Fig.25-4.

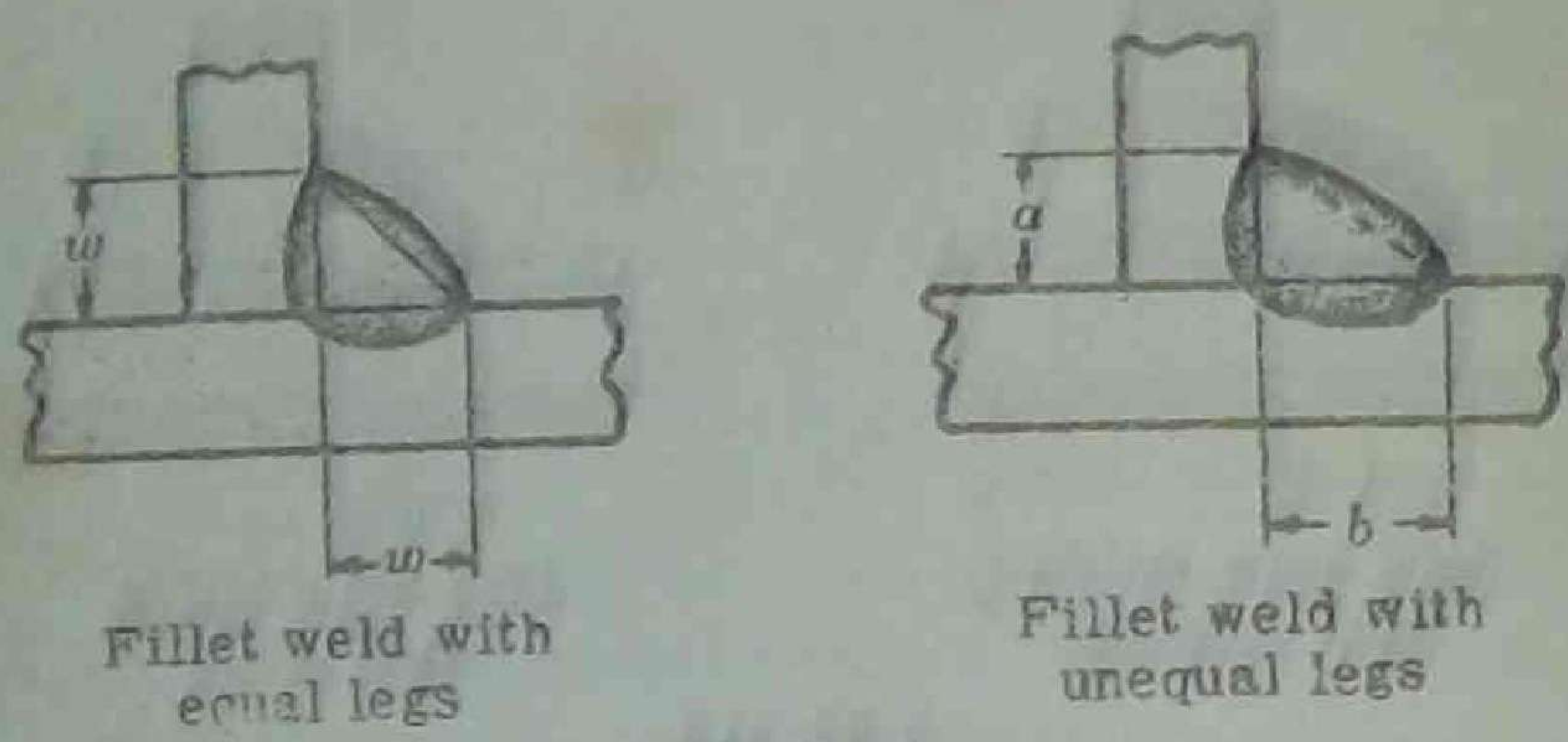


Fig. 25-4

The throat dimension t for a fillet weld having equal legs is obtained by multiplying the size of the fillet weld (the leg length) by 0.707, i.e. $t = 0.707w$. See Fig.25-5.

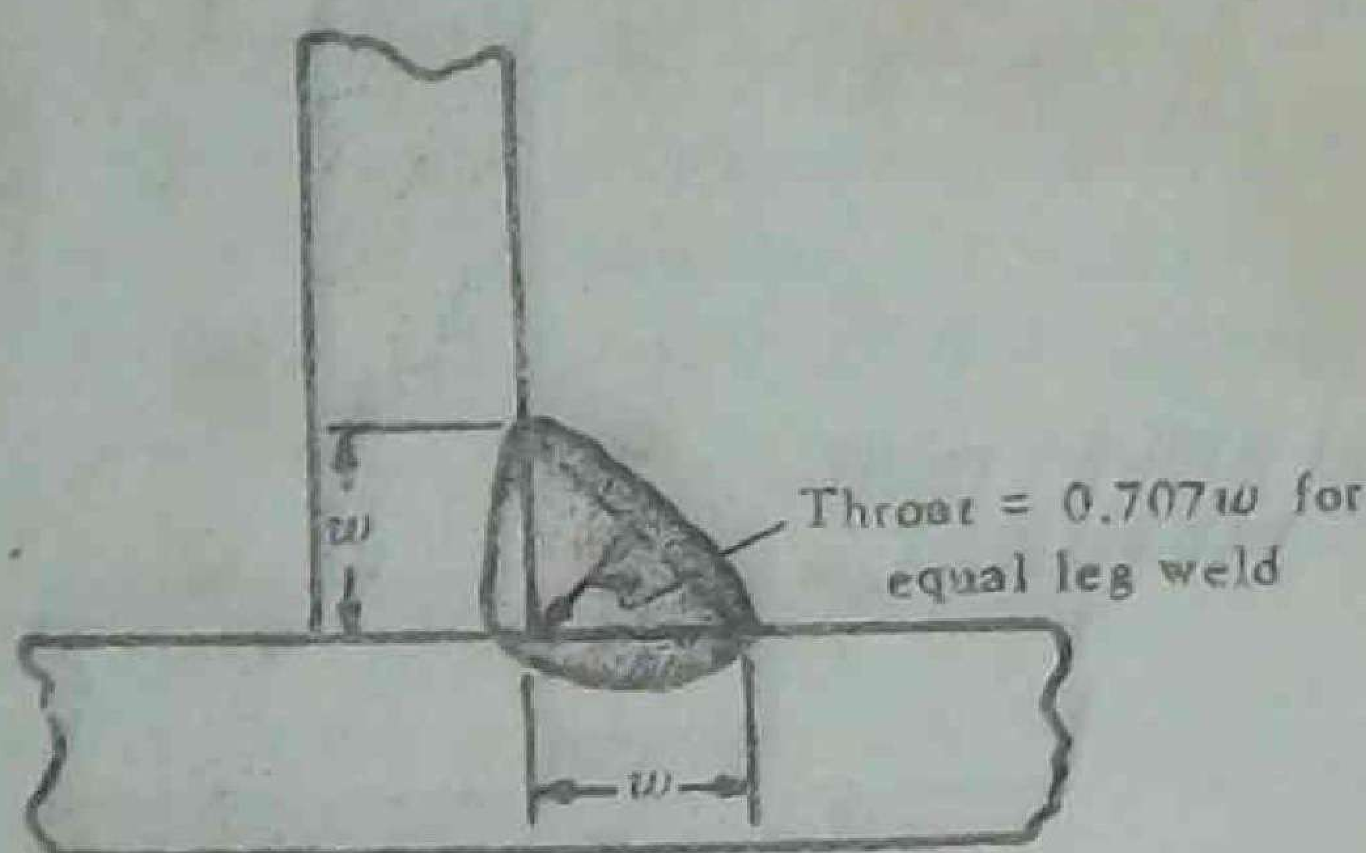


Fig. 25-5

The actual throat t_a obtained with automatic welding is larger than the theoretical throat t (See Fig.25-6). For a penetration of p, the leg length is $(w + p)$, and the throat dimension is $0.707(w + p)$. AWS does not allow for the extra strength due to penetration, although the effect can be taken into account if desired.

Stress in a fillet weld shall be considered as shear stress on the throat for any direction of applied load.

The allowable parallel load per inch of weld in a statically loaded fillet weld is

$$F_{all} = s_{all}A = 13{,}600(0.707w) = 9600w$$

where s_{all} = allowable shear stress = 13,600 psi, according to AWS Code.
A = throat area of 1" of weld at 45°, which is 0.707w
w = leg size, inches.

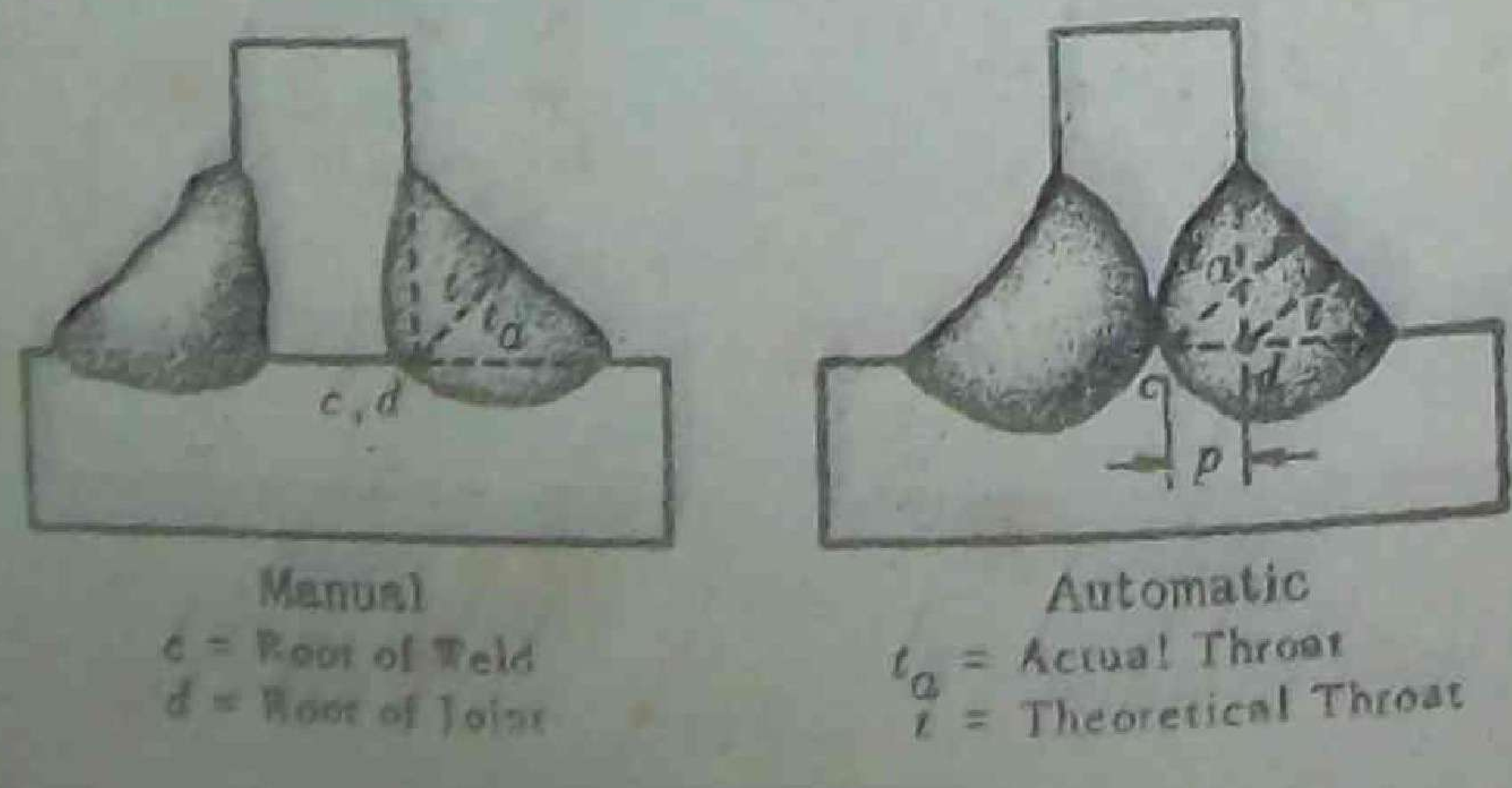
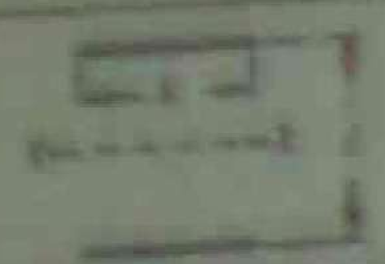
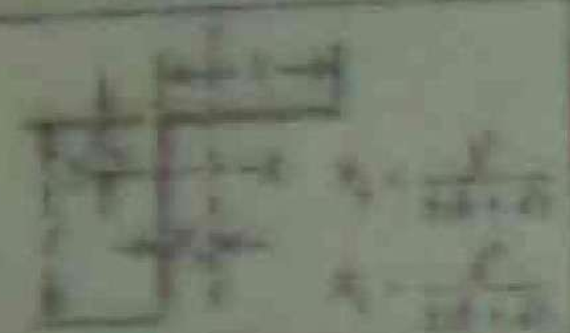
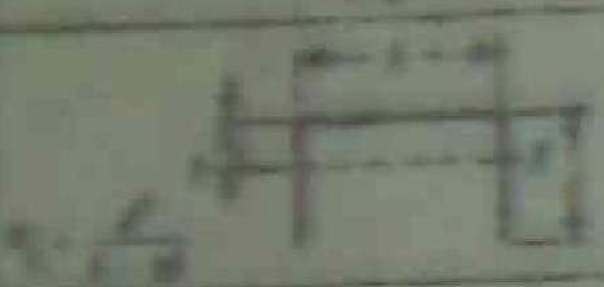
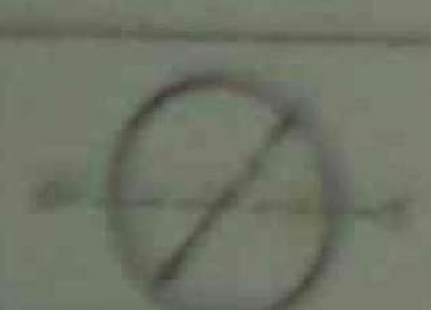
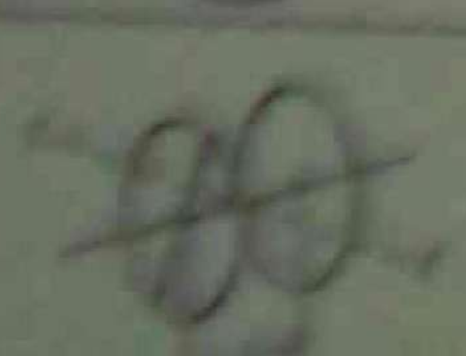
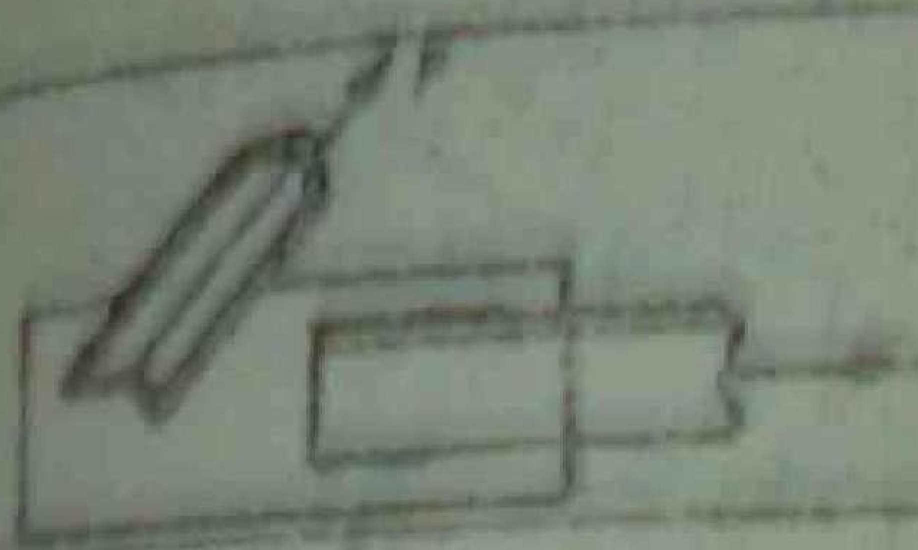
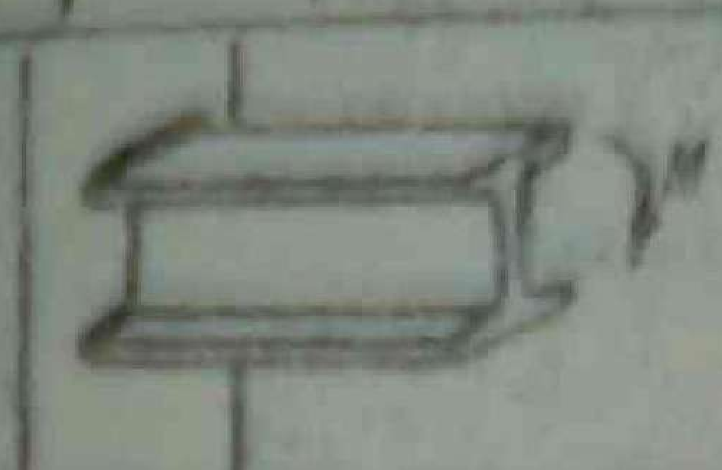


Fig. 25-6

Properties of Weld Treated as a Line		
Outline of Welded Joint b = width d = depth	Bending (about horizontal axis X-X)	Twisting
	$Z_w = \frac{d^2}{6}$	$J_w = \frac{d^3}{12}$
	$Z_w = \frac{d^2}{3}$	$J_w = \frac{d(3b^2+d^2)}{6}$
	$Z_w = bd$	$J_w = \frac{b^3+3bd^2}{6}$
	$Z_w = \frac{4bd+d^2}{6}$ (top) $= \frac{d^2(4bd+d)}{6(2b+d)}$ (bottom)	$J_w = \frac{(b+d)^4-6b^2d^2}{12(b+d)}$
	$Z_w = bd + \frac{d^2}{6}$	$J_w = \frac{(2b+d)^3}{12} - \frac{b^2(b+d)^2}{(2b+d)}$
	$Z_w = \frac{2bd+d^2}{3}$ (top) $= \frac{d^2(2b+d)}{3(b+d)}$ (bottom)	$J_w = \frac{(b+2d)^3}{12} - \frac{d^2(b+d)^2}{(b+2d)}$
	$Z_w = bd + \frac{d^2}{3}$	$J_w = \frac{(b+d)^3}{6}$
	$Z_w = \frac{2bd+d^2}{3}$ (top) $= \frac{d^2(2b+d)}{3(b+d)}$ (bottom)	$J_w = \frac{(b+2d)^3}{12} - \frac{d^2(b+d)^2}{(b+2d)}$
	$Z_w = \frac{4bd+d^2}{3}$ (top) $= \frac{4bd^2+d^3}{6b+3d}$ (bottom)	$J_w = \frac{d^3(4b+d)}{6(b+d)} + \frac{b^3}{6}$
	$Z_w = bd + \frac{d^2}{3}$	$J_w = \frac{b^3+3bd^2+d^3}{6}$
	$Z_w = 2bd + \frac{d^2}{3}$	$J_w = \frac{2b^3+6bd^2+d^3}{6}$
	$Z_w = \frac{\pi d^2}{4}$	$J_w = \frac{\pi d^3}{4}$
	$Z_w = \frac{\pi d^2}{2} + \pi D^2$	

Courtesy The Lincoln Electric Co.

Fig. 25-7 (a)

Type of Loading		Standard Design Formula Stress lb/in²	Treating the Weld as a Line Force lb/in
	Primary Welds transmit entire load		
	tension or compression	$s = \frac{P}{A}$	$f = \frac{P}{A_w}$
	vertical shear	$s = \frac{V}{A}$	$f = \frac{V}{A_w}$
	bending	$s = \frac{M}{S}$	$f = \frac{M}{S_w}$
	twisting	$s = \frac{Tc}{J}$	$f = \frac{Tc}{J_w}$
	Secondary Welds hold section together – low stress		
	horizontal shear	$s = \frac{Vay}{It}$	$f = \frac{Vay}{In}$
	torsional horizontal shear	$s = \frac{Tt}{J}$	$f = \frac{Tt}{J}$

Courtesy The Lincoln Electric Co.

Fig. 25-7 (b)

(c) **Torsion only.**

In this case the critical points occur all along the outer surface of the member.

$s_x = 0$

$\tau_{xy} = \dfrac{Tr}{J} = \dfrac{(2000)(1)(32)}{\pi 2^4} = 1272$ psi

$s_n(\text{max}) = +1272$ psi (tension)

$s_n(\text{min}) = -1272$ psi (compression)

$\tau(\text{max}) = 1272$ psi (shear)

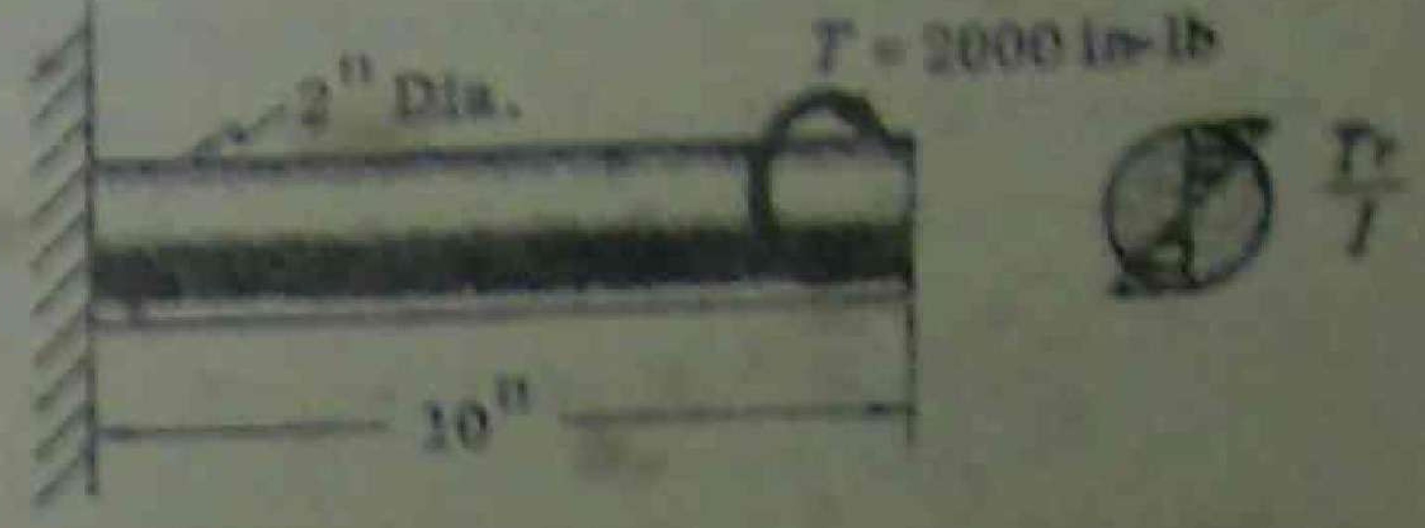


Fig. 2-6

(d) **Bending and torsion.**

Points A and B are critical.

$s_x = +Mc/I = +7650$ psi at point A

$s_x = -7650$ psi at point B

$\tau_{xy} = Tr/J = 1272$ psi at points A and B

$s_n(\text{max}) = +7650/2 + \sqrt{(7650/2)^2 + (1272)^2}$

$= +3825 + 4030 = +7855$ psi (tension at point A)

$s_n(\text{min}) = +3825 - 4030 = -205$ psi (compression at point A)

$s_n(\text{max}) = -3825 + 4030 = +205$ psi (tension at point B)

$s_n(\text{min}) = -3825 - 4030 = -7855$ psi (compression at point B)

$\tau(\text{max}) = \dfrac{+7855-(-205)}{2} = +4030$ psi (shear at point A)

$\tau(\text{max}) = \dfrac{-7855 - 205}{2} = -4030$ psi (shear at point B)

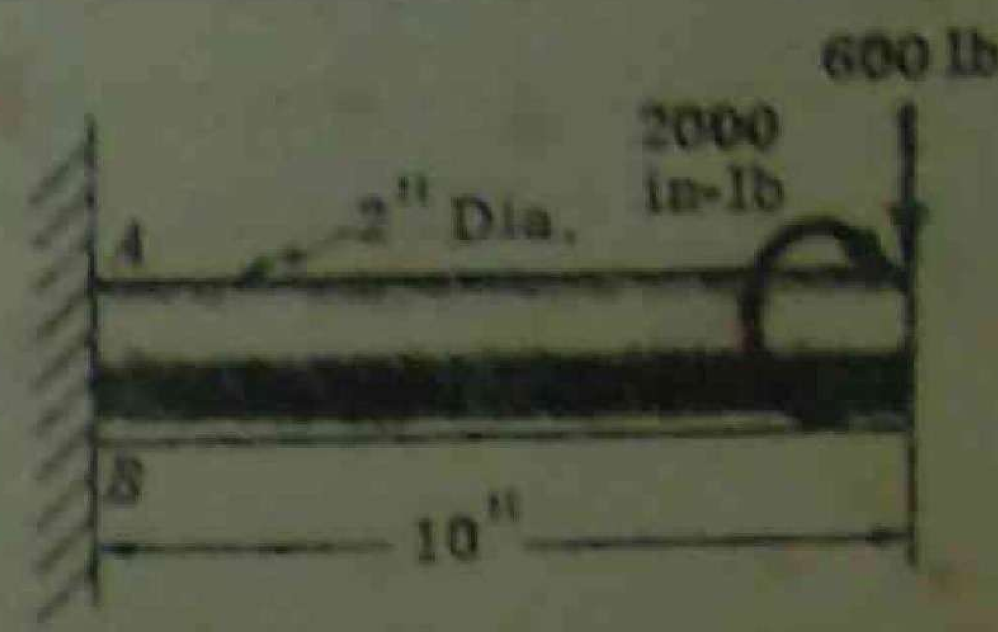


Fig. 2-7

Note that the magnitudes of the stresses at points A and B are the same. The signs of the maximum normal stresses indicate tension or compression, while the sign of the maximum shear stress is of no consequence since design is based on the magnitude.

(e) **Bending and axial load.**

$\tau_{xy} = 0$ at the critical points A and B.

At point A:

$s_x = +P/A + Mc/I = +954 + 7650 = +8604$ psi (tension)

$s_n(\text{max}) = s_x = +8604$ psi (tension)

$s_n(\text{min}) = 0$

$\tau(\text{max}) = \frac{1}{2}(8604) = 4302$ psi (shear)

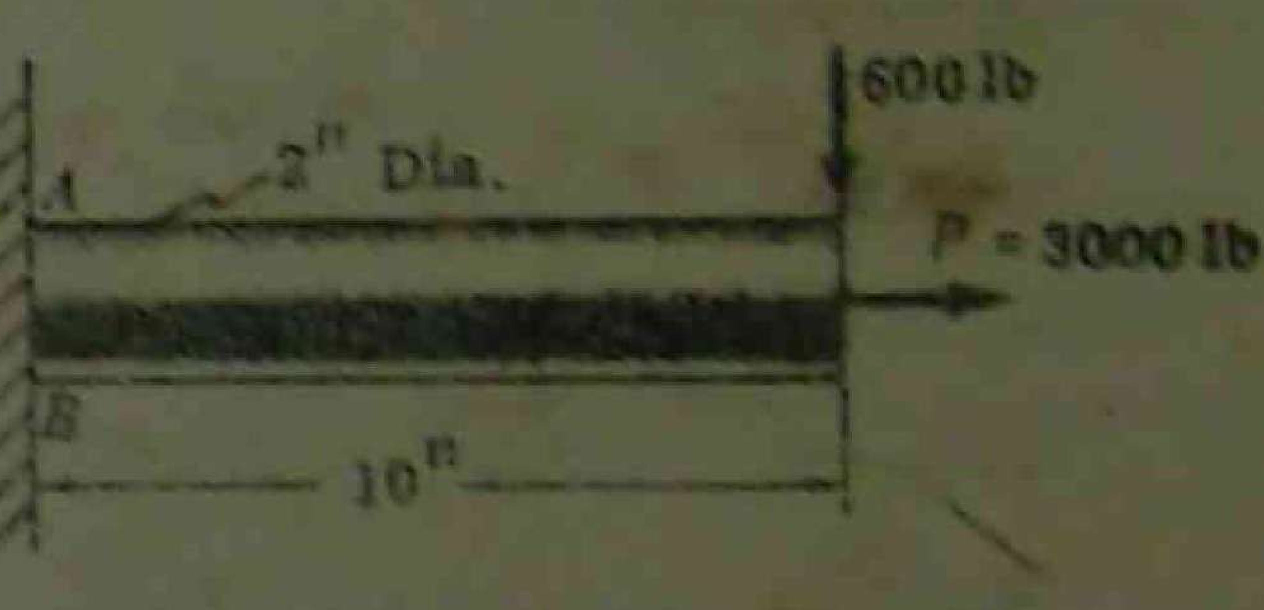


Fig. 2-8

At point B:

$s_x = +P/A - Mc/I = +954 - 7650 = -6696$ psi (compression)

$s_n(\text{max}) = 0$

$s_n(\text{min}) = -6696$ psi (compression)

$\tau(\text{max}) = \frac{1}{2}(6696) = 3348$ psi (shear)

(f) **Torsion and axial load.**

The critical points are the points on the outer surface of the member.

$s_x = +P/A = +954$ psi

$\tau_{xy} = Tr/J = 1272$ psi

$s_n(\text{max}) = +954/2 + \sqrt{(954/2)^2 + (1272)^2}$

$= +477 + 1360 = +1837$ psi (tension)

$s_n(\text{min}) = +477 - 1360 = -883$ psi (compression)

$\tau(\text{max}) = 1360$ psi (shear)

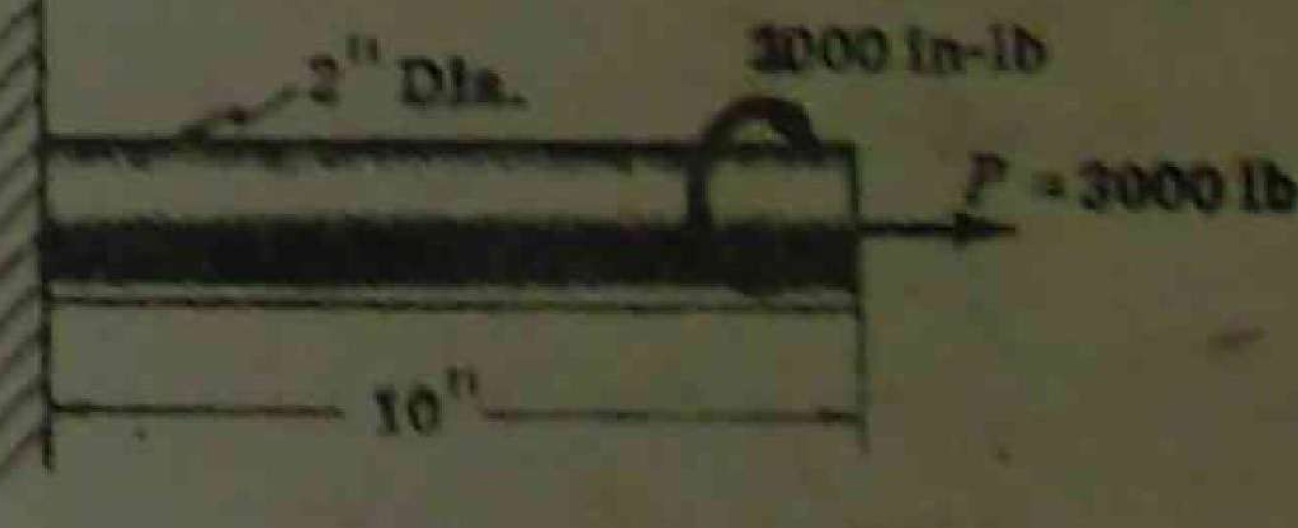


Fig. 2-9

$$s_{s(max)} = \frac{(F \sin 67\tfrac{1}{2}^\circ)(\cos 67\tfrac{1}{2}^\circ + \sin 67\tfrac{1}{2}^\circ)}{wL} = \frac{1.21F}{wL} = \frac{0.924F}{tL}$$ where t is the throat at $67\tfrac{1}{2}^\circ$

(g) Allowable force per inch of weld is $F = \frac{s_{s(all)}wL}{1.21} = \frac{13{,}600w(1)}{1.21} = 11{,}300w$, where w is the leg size.

3. How is the loading distributed in welds with parallel loading if the welds are relatively long?

Solution:

If the welds are long, the loading is not distributed uniformly. The maximum loading per inch of weld depends on the length of the weld. Values of allowable load per inch of weld should be reduced to about 90% that for short welds.

4. Treating the weld as a line, determine the section modulus Z_w in bending of a weld d inches high. Refer to Fig. 25-10.

Solution:

Moment of inertia $I = \int_{-\frac{1}{2}d}^{+\frac{1}{2}d} y^2\,dy = \frac{d^3}{12}$ and $Z_w = \frac{I}{d/2} = \frac{d^2}{6}$.

5. Treating the weld as a line, determine the moment of inertia J_w about the center.

Solution:

Referring to Fig. 25-10, $J_w = \int_{-\frac{1}{2}d}^{+\frac{1}{2}d} y^2\,dy = \frac{d^3}{12}$.

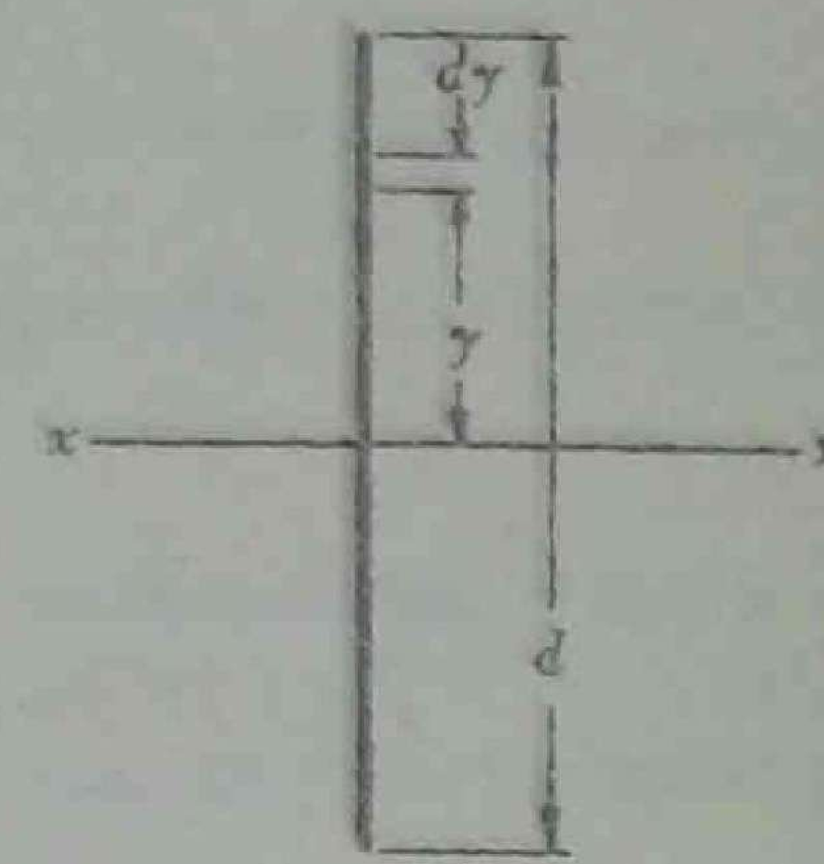


Fig. 25-10

6. Treating the weld as a line, determine the section modulus Z_w about the x-x axis. Refer to Fig. 25-11.

Solution:

From Prob. 4, the moment of inertia of the vertical lines about the x-x axis is $I_1 = 2(d^3/12) = d^3/6$.

The moment of inertia of the horizontal lines is $I_2 = 2[b(d/2)^2] = bd^2/2$.

The total moment of inertia about the x-x axis is $I = I_1 + I_2 = (d^3/6) + (bd^2/2)$.

The section modulus $Z_w = \frac{I}{c} = \frac{(d^3/6) + (bd^2/2)}{d/2} = \frac{d^2}{3} + bd$.

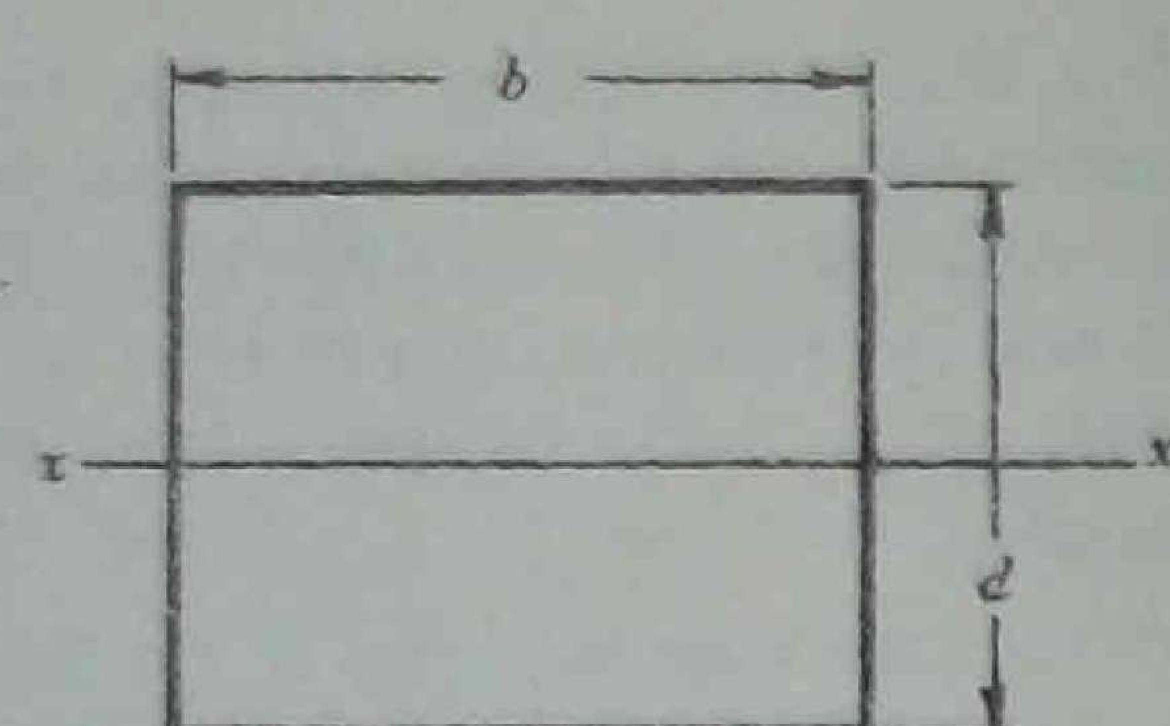


Fig. 25-11

7. Treating the weld as a line, determine the moment of inertia J_w about the center of gravity. Refer to Fig. 25-12.

Solution:

Consider each line separately, determine the effect of each, and add the parts.

Consider the top line, with a differential element dx. The integral of the product of the length of the element and the square of the variable distance to the center of gravity is

$$J_{w_1} = \int r^2\,dx = 2\int_0^{\frac{1}{2}b} [(d/2)^2 + x^2]\,dx$$

$$= 2(\tfrac{d}{2})^2(\tfrac{b}{2}) + \tfrac{2}{3}(\tfrac{b}{2})^3 = \frac{d^2 b}{4} + \frac{b^3}{12}$$

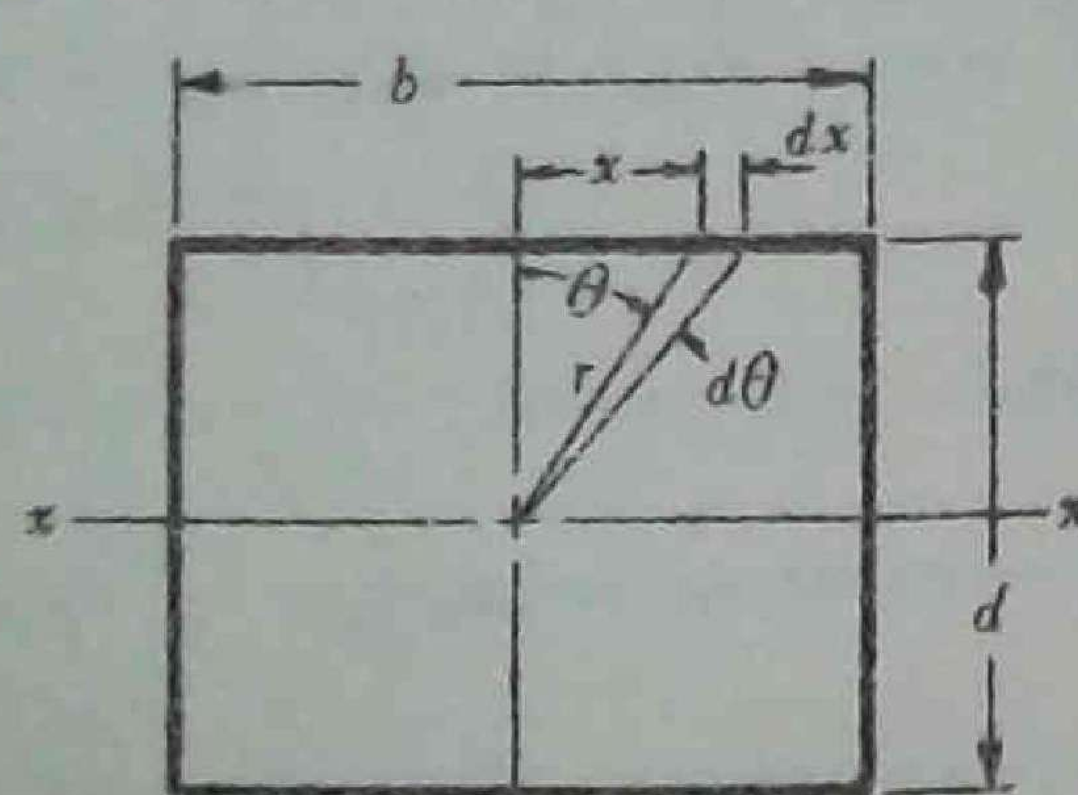


Fig. 25-12

The contribution by the bottom line is the same: $J_{w_2} = \frac{d^2 b}{4} + \frac{b^3}{12}$.

By analogy, the polar moment of inertia of each vertical line is $J_{w_3} = J_{w_4} = \frac{db^2}{4} + \frac{d^3}{12}$.

8. Determine the required fillet weld size for the bracket shown in Fig. 25-13 below.

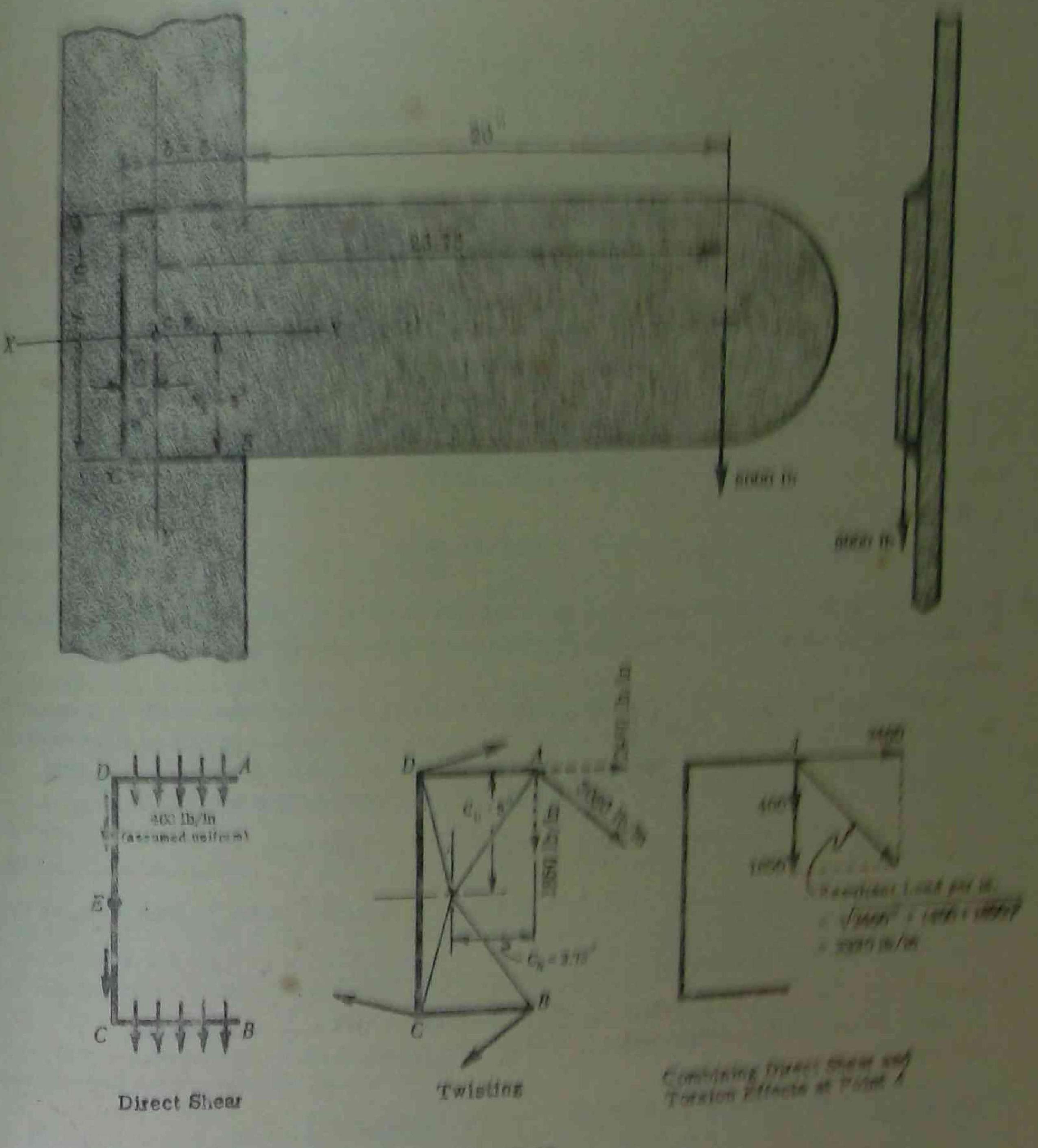


Fig. 25-13

Solution:

(*a*) Determine the center of gravity of the weld, treating the weld as a line with no thickness. See Fig. 25-7(c).

By symmetry, $N_x = 5''$. $N_y = \frac{b^2}{2b + d} = \frac{5^2}{2(5) + 10} = 1.25''$

(*b*) Replace the original 8000 lb force by a force of 8000 lb at the c.g. and a couple = 8000(23.75) = 190,000 in-lb (causing twisting).

(*c*) The vertical force of 8000 lb is assumed uniformly distributed over the weld and causes a loading of (8000)/(5 + 10 + 5) = 400 lb per inch of weld.

(*d*) Now determine the effect of the twisting couple. The polar moment of inertia of the weld, treating it as a line, is

$$I_w = \frac{(2b+d)^3}{12} - \frac{b^2(b+d)^2}{2b+d} = \frac{(10+10)^3}{12} - \frac{5^2(5+10)^2}{2(5)+10} = 385.4 \text{ in}^3$$

(*e*) At points *A* and *B*, the maximum loading *f* from twisting is

$$f = \frac{TC}{I_w} = \frac{(190,000)\sqrt{5^2+3.75^2}}{385.4} = 3080 \text{ lb/in}$$

where C = distance from center of gravity to point being analyzed.

The vertical component $f_v = \frac{3.75}{\sqrt{5^2+3.75^2}}(3080) = 1850$ lb/in.

The horizontal component $f_h = \frac{5}{\sqrt{5^2+3.75^2}}(3080) = 2460$ lb/in.

Note that f_v and f_h can be obtained directly by using the horizontal and vertical distances $C_h = 3.75''$ and $C_v = 5''$ in $f = TC/I_w$.

(*f*) Combining the horizontal and vertical components at point *A*, the loading per inch is

$$F = \sqrt{(2460)^2 + (400+1850)^2} = 3330 \text{ lb/in}$$

(*g*) For steady loads, the weld size is $w = \frac{f_{actual}}{f_{allowable}} = \frac{3330}{9600} = 0.347''$. Use a $\frac{3}{8}''$ weld.

Note that the allowable loading is 9600 lb per inch of weld, the allowable loading for parallel loading being used where there is a combination of transverse and parallel loading.

(*h*) An alternate analysis, which is justifiable on the basis of the distribution of the transverse shear force as used in beam analysis, is to consider the top and bottom welds as carrying no transverse shear force. The maximum transverse shear stress in a rectangular section is $3V/2A$ at the neutral axis. Thus the direct shear at point *A* is zero and at point *E* the maximum shearing force per inch is

$$f_s = \frac{3}{2}\left(\frac{V}{L_w}\right) = \frac{3}{2}\left(\frac{8000}{10}\right) = 1200 \text{ lb/in}$$

At point *A* the resultant force per inch of weld, due to torsion alone, is 3080 lb/in, as determined in (*e*) above.

The size of weld is critical at point *A* (and point *B*) and is

$$w = \frac{f_{actual}}{f_{allowable}} = \frac{3080}{9600} = 0.321''. \quad \text{Use a } \frac{5}{16}'' \text{ or } \frac{3}{8}'' \text{ weld.}$$

9. A circular bar is welded to a steel plate. The bar diameter $d = 2$ in. Determine the size of weld required.

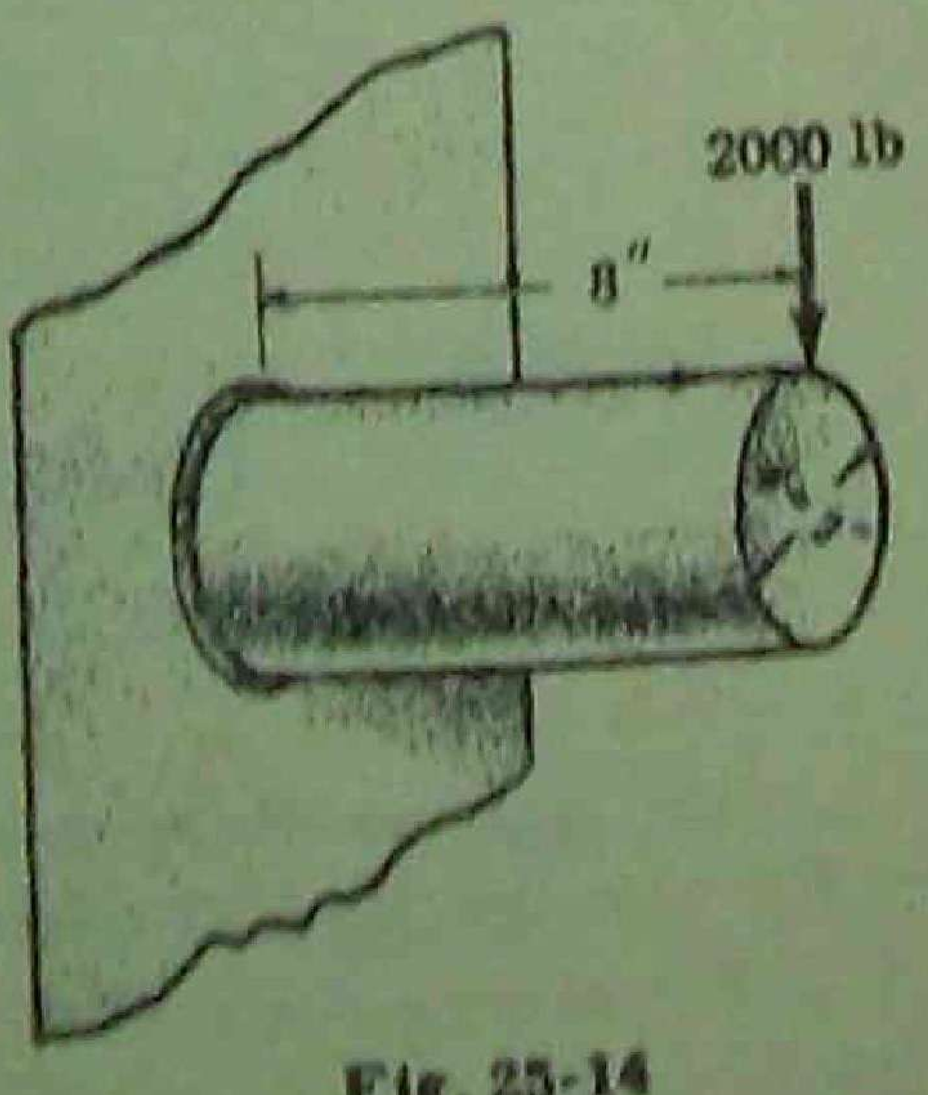


Fig. 25-14

Solution:

Bending moment = 2000(8) = 16,000 in-lb. Shearing force = 2000 lb.

Section modulus of weld treated as a line is [see Fig. 25-7(*a*)]

$$Z_w = \tfrac{1}{4}\pi d^2 = \tfrac{1}{4}\pi(2)^2 = 3.14 \text{ in}^2$$

Force per inch of weld, at top and bottom, is

$$f_b = M/Z_w = 16,000/3.14 = 5100 \text{ lb/in}$$

Vertical shear, assuming uniform distribution of the shear force, is

$$f_s = V/L_w = 2000/2\pi = 318 \text{ lb/in}$$

Resultant load $f = \sqrt{5100^2 + 318^2} = 5110$ lb/in.

Size of weld $w = \frac{f_{actual}}{f_{allowable}} = \frac{5110}{9600} = 0.53''$. A $\frac{1}{2}''$ weld should be satisfactory.

10. A plate girder is fabricated by welding. What size of fillet welds to join the flanges to the web is required for a transverse load (shear force) of 150,000 lb applied at the section under consideration? Refer to Fig. 25-15.

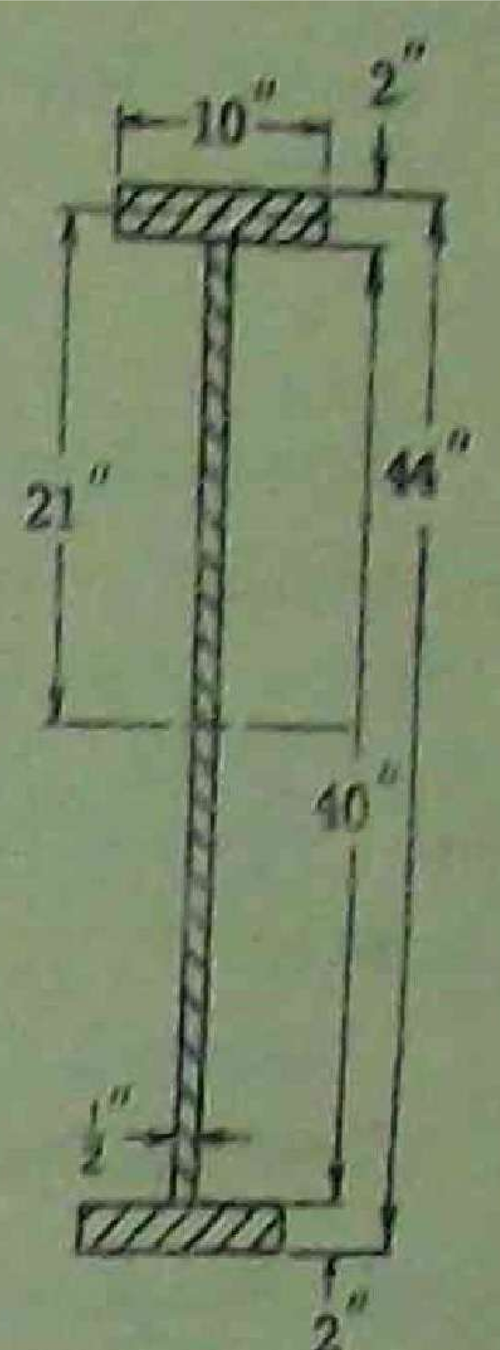


Fig. 25-15

Solution:

The weld required at the junction of the web and flange is considered as a secondary weld, inasmuch as it is required to hold the parts together. The deflection of the beam is not affected significantly even if the weld is omitted.

The loading per inch of weld at the junction of the web and flange is

$$f = \frac{VAY}{In} = \frac{(150,000)(20)(21)}{(20,020)(2)} = 1575 \text{ lb/in}$$

where V = shear force = 150,000 lb
A = area of section above the weld = 10(2) = 20 in^2
Y = distance to the center of gravity of the area above the weld (= 21 in.)
I = moment of inertia of the whole section about c.g. axis of I-beam (= 20,020 in^4)
n = number of welds (= 2).

The weld leg size w = 1575/9600 = 0.164" (leg size of continuous weld).

Even though there may be little or no stress on some welds, for practical reasons it is best not to put too small a weld on a thick plate. The adjacent table, given by American Welding Society, can be used as a guide. Thus the minimum weld size as calculated is 0.164" for a continuous weld. However, by the adjacent table, the minimum weld is $\frac{3}{8}$" for a 2" plate. Note that the leg size of the fillet weld need not exceed the thickness of the thinner plate.

Thickness of thicker plate	Minimum weld size
up to $\frac{1}{2}$"	3/16 in.
over $\frac{1}{2}$" up to $\frac{3}{4}$"	1/4 in.
over $\frac{3}{4}$" up to $1\frac{1}{4}$"	5/16 in.
over $1\frac{1}{4}$" up to 2"	3/8 in.
over 2" up to 6"	1/2 in.
over 6"	5/8 in.

Because of the greater strength of the $\frac{3}{8}$" weld, intermittent welds can be used. Lincoln Electric Co. recommends, also, that the size of the fillet weld used for design calculations or determination of length must not exceed $\frac{2}{3}$ of the web thickness, or $\frac{2}{3}(\frac{1}{2})$ = 0.333". This recommendation is based on limiting the shear stress in the thinner plate to 13,000 psi (as given by the AWS Bridge Code). Thus, even though a $\frac{3}{8}$" weld is to be used, the calculations will be based on a weld 0.333".

$$R = \frac{\text{size of continuous weld needed}}{\text{size of intermittent weld used}} = \frac{0.164}{0.333} = 49\%$$

From the table for percent of continuous weld, using the value for 50%, the length of intermittent weld and spacing can be 2-4, 3-6, or 4-8.

A final recommendation, then, for the weld is a $\frac{3}{8}$" weld (leg size), two inches long, on 4 inch centers.

11. A rectangular beam is to be welded to a plate. The maximum load of 3000 lb is applied repetitiously. Determine the size of weld required for 10,000,000 cycles. Assume the shear load is distributed uniformly over the entire weld. Refer to Fig. 25-16 below.

Solution:

Consider the horizontal welds where the bending stress is maximum (the top and bottom welds are stressed the same).

The bending moment varies from a maximum of 3000(6) = 18,000 in-lb in one direction to a maximum of 3000(6) = 18,000 in-lb in the opposite direction. The shear force varies from 3000 lb up to 3000 lb down.

The section modulus of the weld is $Z_w = bd + d^2/3 = (2)(3) + 3^2/3 = 9 \text{ in}^2$

The load in lb/in due to bending is $f = M/Z_w = 18,000/9 = 2000$ lb/in

$$\text{Average shear force} = \frac{V}{L_w} = \frac{3000}{2+3+2+3} = 300 \text{ lb/in}$$

$$\text{Maximum force/in} = \sqrt{2000^2 + 300^2} = 2020 \text{ lb/in.}$$

The maximum force varies from 2020 lb/in in one direction to 2020 lb/in in the opposite direction. This is true for both top and bottom welds.

The allowable force/in for 2,000,000 cycles is

$$f_{2,000,000} = \frac{5090}{1 - \frac{1}{2}K} = \frac{5090}{1 - \frac{1}{2}(-1)} = 3390 \text{ lb/in}$$

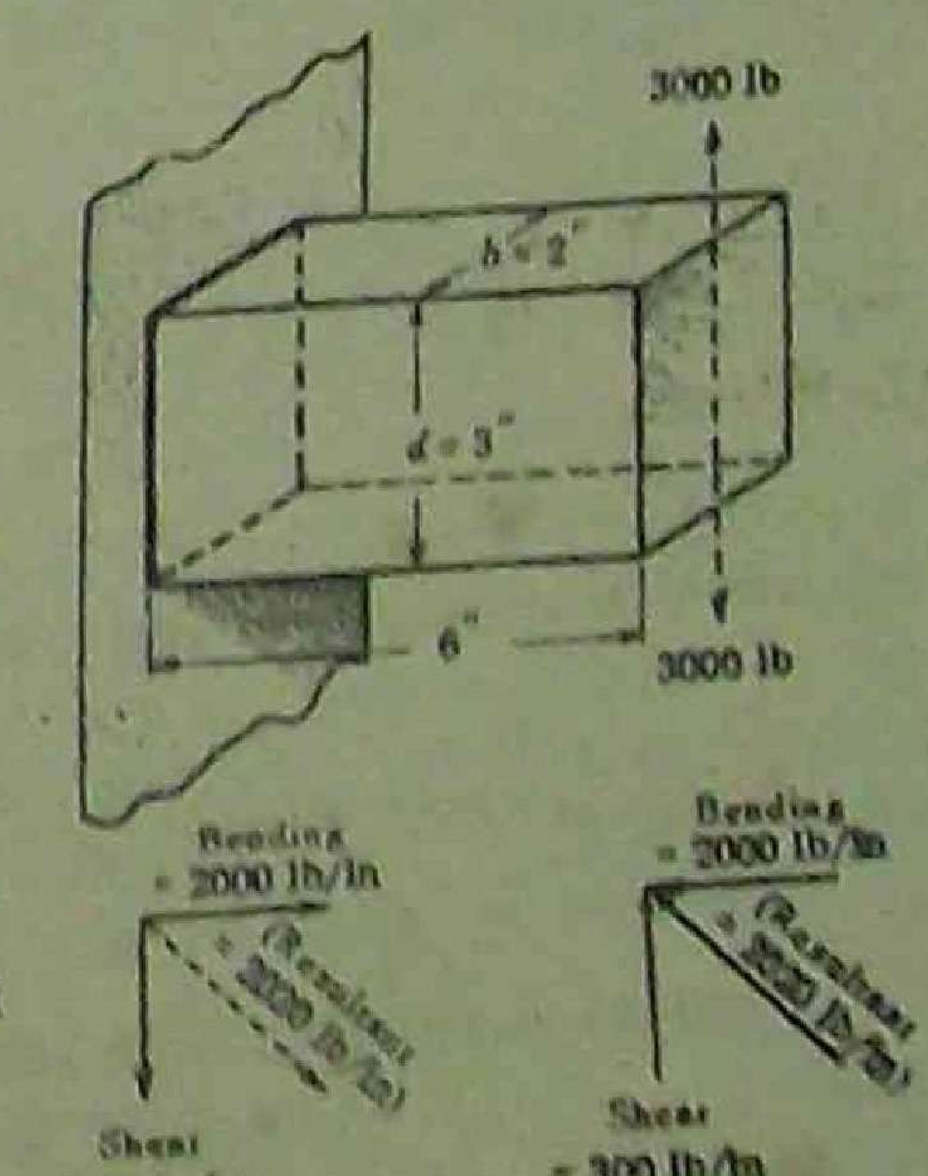


Fig. 25-16

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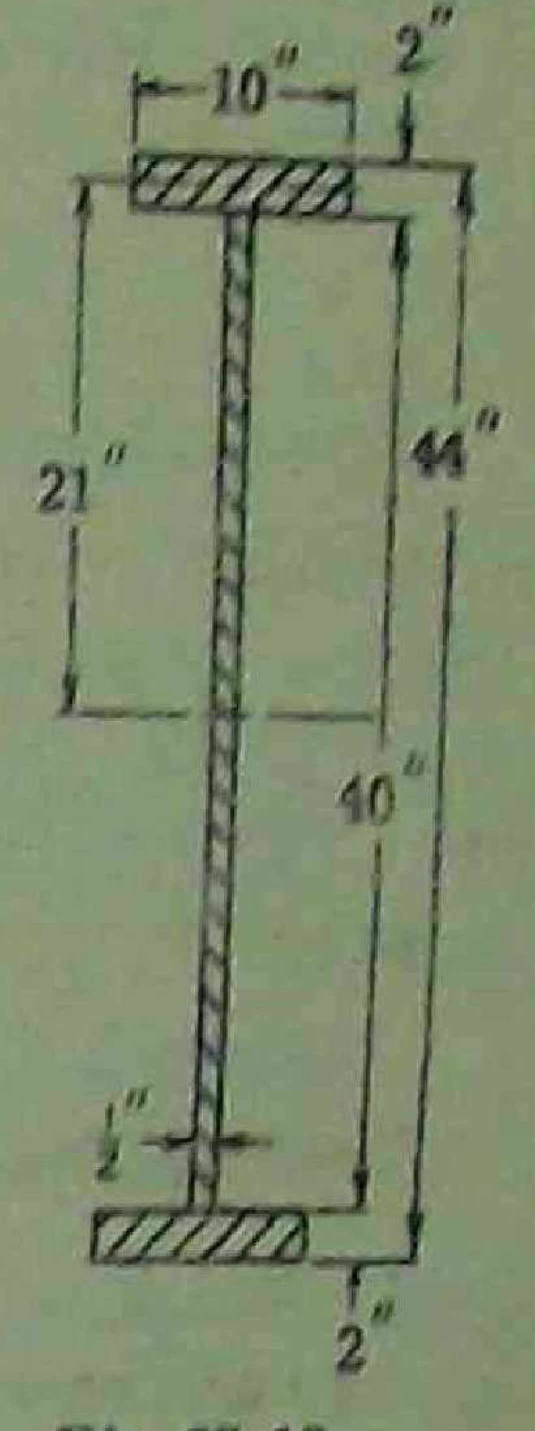

Fig. 25-15

Solution:

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where V = shear force = 150,000 lb
A = area of section above the weld = 10(2) = 20 in^2
Y = distance to the center of gravity of the area above the weld (= 21 in.)
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Even though there may be little or no stress on some welds, for practical reasons it is best not to put too small a weld on a thick plate. The adjacent table, given by American Welding Society, can be used as a guide. Thus the minimum weld size as calculated is 0.164" for a continuous weld. However, by the adjacent table, the minimum weld is $\frac{3}{8}$" for a 2" plate. Note that the leg size of the fillet weld need not exceed the thickness of the thinner plate.

Thickness of thicker plate	Minimum weld size
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over 2" up to 6"	1/2 in.
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From the table for percent of continuous weld, using the value for 50%, the length of intermittent weld and spacing can be 2-4, 3-6, or 4-8.

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Solution:

Consider the horizontal welds where the bending stress is maximum (the top and bottom welds are stressed the same).

The bending moment varies from a maximum of 3000(6) = 18,000 in-lb in one direction to a maximum of 3000(6) = 18,000 in-lb in the opposite direction. The shear force varies from 3000 lb up to 3000 lb down.

The section modulus of the weld is $Z_w = bd + d^2/3 = (2)(3) + 3^2/3 = 9 \text{ in}^2$

The load in lb/in due to bending is $f = M/Z_w = 18{,}000/9 = 2000$ lb/in

$$\text{Average shear force} = \frac{V}{L_w} = \frac{3000}{2+3+2+3} = 300 \text{ lb/in}$$

$$\text{Maximum force/in} = \sqrt{2000^2 + 300^2} = 2020 \text{ lb/in.}$$

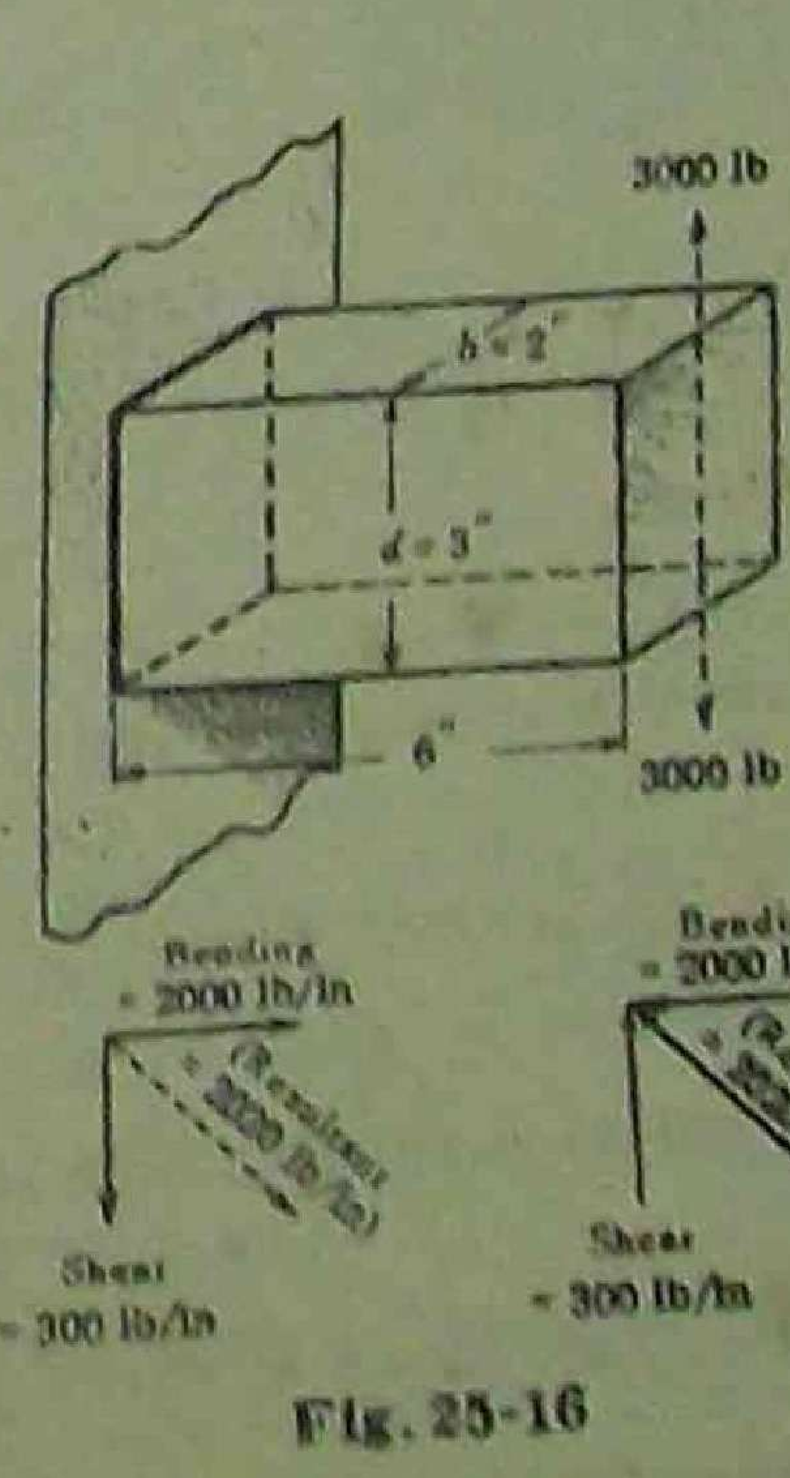

Fig. 25-16

The maximum force varies from 2020 lb/in in one direction to 2020 lb/in in the opposite direction. This is true for both top and bottom welds.

The allowable force/in for 2,000,000 cycles is

$$f_{2,000,000} = \frac{5090}{1-\frac{1}{2}K} = \frac{5090}{1-\frac{1}{2}(-1)} = 3390 \text{ lb/in}$$

SUPPLEMENTARY PROBLEMS

6. A gas engine develops 80 indicated horsepower at 1800 rpm mean speed. The maximum variation of energy per revolution is 27% of the mean energy, and the allowable coefficient of speed fluctuation is 0.02. Assume that the rim provides 95% of the needed flywheel effect ($K = 0.95$). The mean velocity of the flywheel rim is limited to 8480 ft/min. Determine the mean diameter and the weight of rim required. *Ans.* 18 in., 30.7 lb

7. A gas engine develops 80 indicated horsepower at 1600 rpm mean speed. If the maximum variation of energy per revolution is 27% of the mean energy, and if the allowable coefficient of speed fluctuation is 0.02, determine the necessary weight of flywheel. Assume all the flywheel effect comes from the flywheel. The flywheel is to be a plate of outside diameter 18 in. mounted on the shaft. *Ans.* Weight of plate = 64.6 lb

8. A crusher drive-shaft rotates at maximum speed of 60 rpm and requires an average power input of 10 hp. If the maximum energy variation per cycle is equal to the mean energy of the cycle and if the speed must not drop more than 10% during the crushing operation, determine the required weight of a flywheel rim with a mean diameter of 80 in. The crushing operation occurs in each revolution of the drive-shaft. Assume $K = 0.95$. *Ans.* 4040 lb

9. A cast iron flywheel rotating at 40 rpm maximum is to furnish 75,000 ft-lb of energy to a punch during $\frac{1}{4}$ revolution with a 10% reduction in speed. The maximum velocity at the mean radius of the rim is not to exceed 3000 ft/min. What cross section area of the rim is necessary if 95% of the flywheel effect is produced by the flywheel? Cast iron weighs 0.255 lb/in³. *Ans.* 41.8 in²

10. A single cylinder double acting engine delivers 250 hp at 100 rpm mean speed. The maximum variation of energy per revolution is 10% of the mean energy, and the speed variation is limited to 2% either way from the mean speed. The mean diameter of the rim is 8 ft. Assuming that the hub, spokes, and shaft contribute 5% of the flywheel effect ($K = 0.95$) and that cast iron weighs 0.255 lb/in³, determine (*a*) the coefficient of speed fluctuation, (*b*) the weight of rim, (*c*) the cross section area of the rim. *Ans.* 0.04, 3600 lb, 46.8 in²

STRESSES IN FLYWHEELS. The stress in what is called a free rotating ring is a very simple quick approximation for the stresses in the thin rim of a rotating flywheel. The effect of the spokes is neglected and only the stress due to inertia loading is considered.

(*a*) Consider half a ring isolated as shown in Fig. 26-1. The differential mass dM is

$$dM = \frac{r(d\theta)tb\rho}{g}$$

where r = mean radius, in.

$d\theta$ = differential angle subtended by the differential mass, radians

t = thickness of rim, in.; b = width of rim, in.

ρ = weight density, lb/in³; g = 386 in/sec².

(*b*) The differential inertia load f = (mass)(acceleration) = $\left(\frac{r(d\theta)tb\rho}{g}\right)(r\omega^2)$, where ω = angular velocity in rad/sec.

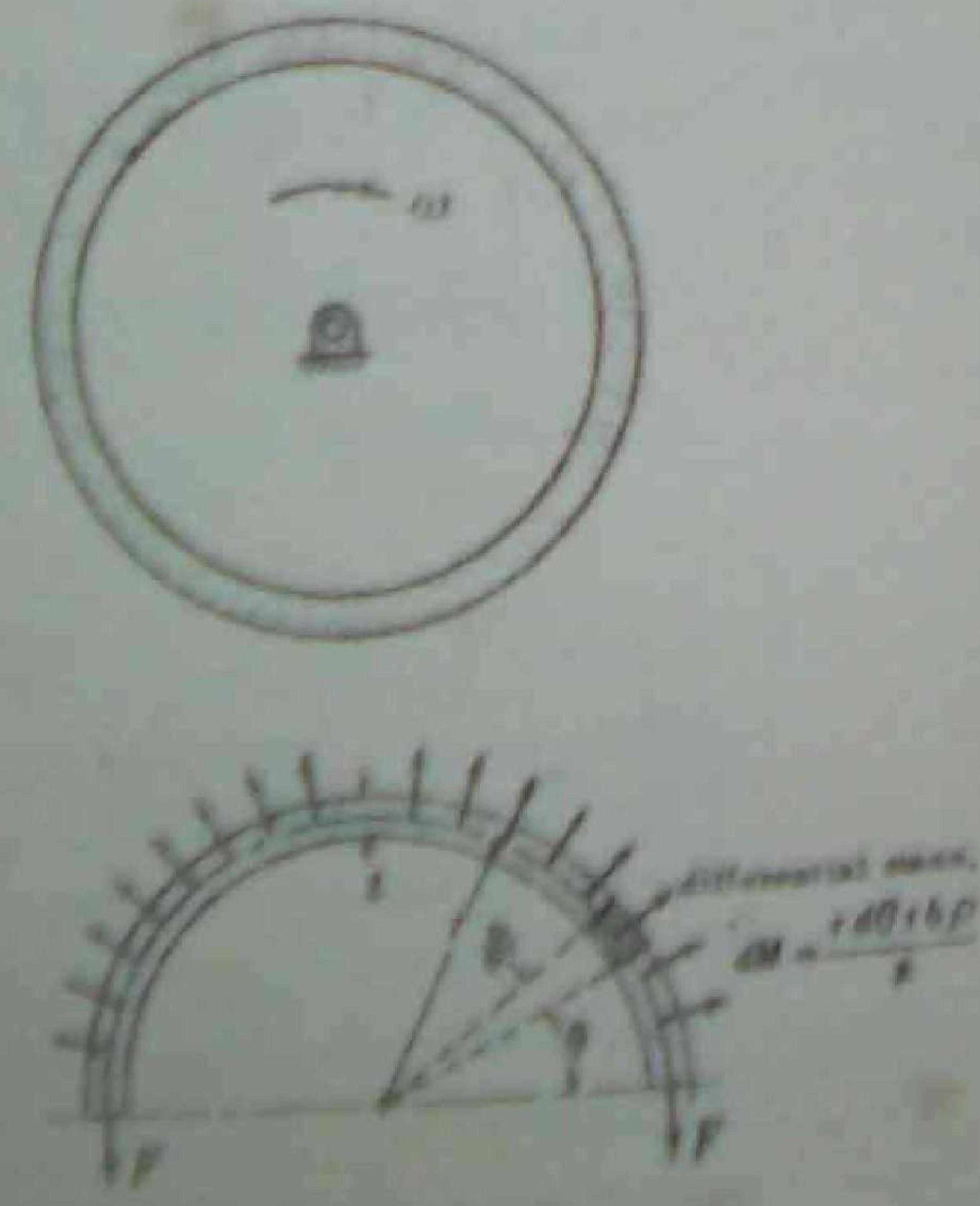

Fig. 26-1

(c) The vertical component of the differential inertia load is $df_y = \left[\frac{r(d\theta)tb\rho}{g}(r\omega^2)\right]\sin\theta$.
Note that horizontal components balance.

(d) The vertical component of the inertia load is balanced by the tensile forces at the cut sections:

$$2F = \int_0^{\pi} \frac{r(d\theta)tb\rho}{g}(r\omega^2)\sin\theta = \frac{r^2tb\rho\omega^2}{g}\left[-\cos\theta\right]_0^{\pi} = \frac{r^2tb\rho\omega^2}{g}(2)$$

(e) Assuming that the tensile stress s_t (psi) is uniformly distributed across a section, then

$$2F = 2(s_t tb) = \frac{r^2tb\rho\omega^2}{g}(2)$$

or

$$s_t = \frac{r^2\rho\omega^2}{g} = \frac{\rho v^2}{g}$$

where $v = r\omega$ is the velocity (in/sec) at the mean radius.

SOLVED PROBLEM

11. Determine the maximum permissible velocity v in a cast iron thin rim of a flywheel if the maximum allowable tensile stress s_t in cast iron is 4000 psi. Cast iron weighs 0.255 lb/in³.

Solution:

$$s_t = \frac{\rho v^2}{g}, \qquad 4000 = \frac{0.255\,v^2}{386}, \qquad v = 2460 \text{ in/sec} = 205 \text{ ft/sec}$$

Note that bending in the rim has been neglected.

SUPPLEMENTARY PROBLEMS

12. Determine the maximum permissible velocity in a steel thin rim of a flywheel if steel weighs 0.283 lb/in³ and the allowable stress is 20,000 psi. *Ans.* 435 ft/sec

13. Considering the cast iron rim of a flywheel as a thin ring and considering the thin rim as a free rotating ring, determine the maximum tensile stress due to rotation. The rim has width 6 in., thickness 4 in., and mean diameter 30 in. The flywheel is rotating at 2000 rpm. Cast iron weighs 450 lb/ft³. *Ans.* 6660 psi

THE MAXIMUM TENSILE STRESS in a thin rotating rim of a flywheel where bending as well as the normal stress due to inertia is considered, is a bit more complicated. An equation developed from those derived by Timoshenko on a rational basis, taking into account the axial force in the spokes, bending, and normal stress, obtained on the basis of treating the rim as a thin ring and neglecting the curvature in the rim is, from $\frac{P}{A} \pm \frac{Mc}{I}$, using the sign which gives the larger value,

$$s_t = \frac{qv^2}{btg}\left[1 - \frac{\cos\phi}{3C\sin\alpha} \pm \frac{2r}{Ct}\left(\frac{1}{\alpha} - \frac{\cos\phi}{\sin\alpha}\right)\right]$$

Fig. [illegible]

where s_t = tensile stress, psi
q = weight of rim per inch of length = $bt\rho$, lb/in
v = velocity at the mean radius, in/sec
b = width of rim, in.; t = thickness of rim, in.; g = 386 in/sec²
ϕ = angle from centerline between spokes to the section where the stress is being found
2α = angle between spokes
r = mean radius of rim, in.
C = a constant depending on the cross section area of the rim, area of a spoke, proportions of the rim, and angle between spokes. C is given by

$$C = \frac{12r^2}{t^2}\left[\frac{1}{2\sin^2\alpha}\left(\frac{\sin 2\alpha}{4} + \frac{\alpha}{2}\right) - \frac{1}{2\alpha}\right] + \frac{1}{2\sin^2\alpha}\left(\frac{\sin 2\alpha}{4} + \frac{\alpha}{2}\right) + \frac{A}{A_1}$$

where A = area of cross section of the rim = bt
A_1 = area of cross section of a spoke.

The numerical values for C for different numbers of spokes, to simplify the arithmetic, are:

$$4 \text{ spokes } (2\alpha = 90^\circ): \quad C = \frac{12r^2}{t^2}(0.00608) + 0.643 + \frac{A}{A_1}$$

$$6 \text{ spokes } (2\alpha = 60^\circ): \quad C = \frac{12r^2}{t^2}(0.00169) + 0.957 + \frac{A}{A_1}$$

$$8 \text{ spokes } (2\alpha = 45^\circ): \quad C = \frac{12r^2}{t^2}(0.00076) + 1.274 + \frac{A}{A_1}$$

The axial force F in each spoke is $F = \dfrac{2qv^2}{3gC}$ lb.

SOLVED PROBLEM

14. (a) Determine the maximum tensile stress in the thin rim of a steel flywheel rotating at 600 rpm (20π rad/sec). The mean radius of the rim is 60" (r = 60"). The flywheel rim is 8" thick (t = 8") and 12" wide (b = 12"). The area of a cross section of the rim is $A = bt = 96\text{ in}^2$. Each of the six spokes is constant in cross section with a cross section area $A_1 = 16\text{ in}^2$. (Find the maximum tensile stress in the rim at two sections, $\phi = 30^\circ$ and $\phi = 0^\circ$.)

(b) Compare the stress determined with that of a free rotating ring.

(c) Calculate the axial stress in each spoke.

Solution:

(a) For 6 spokes, $C = \dfrac{12r^2}{t^2}(0.00169) + 0.957 + \dfrac{A}{A_1} = \dfrac{12(60)^2}{8^2}(0.00169) + 0.957 + \dfrac{96}{16} = 8.10.$

At the section of the rim where the spoke is located, $\phi = 30^\circ$. The stress at this section is

$$s_t = \frac{qv^2}{bg}\left[1 - \frac{\cos\phi}{3C\sin\alpha} \pm \frac{2r}{Ct}\left(\frac{1}{\alpha} - \frac{\cos\phi}{\sin\alpha}\right)\right]$$

$$= \frac{27.2(3770)^2}{12(8)(386)}\left[1 - \frac{\cos 30^\circ}{3(8.10)\sin 30^\circ} \pm \frac{2(60)}{8.10(8)}\left(\frac{1}{\pi/6} - \frac{\cos 30^\circ}{\sin 30^\circ}\right)\right]$$

$$= 10{,}400[1 - 0.0714 \pm 1.85(1.91 - 1.74)]$$

$= 12,900$ psi using the + sign for the maximum

where $q = 96(12)(0.283) = 27.2$ lb/in, $v = r\omega = 3770$ in/sec, $2\alpha = 60^\circ$ for 6 spokes.

At the section of the rim midway between spokes, $\phi = 0^\circ$. The stress at this section is

$$s_t = \frac{27.2(3770)^2}{12(8)(386)}\left[1 - \frac{1}{3(8.10)\sin 30^\circ} \pm \frac{2(60)}{8.10(8)}\left(\frac{1}{\pi/6} - \frac{1}{\sin 30^\circ}\right)\right]$$

$$= 10{,}400[1 - 0.0824 \pm 1.85(1.91 - 2)]$$

$= 11,300$ psi using the − sign for the maximum.

The maximum tensile stress in the rim occurs at the section where the spoke is located and is 12,900 psi.

(b) The stress in a free rotating ring $= \dfrac{\rho v^2}{g} = \dfrac{0.283(3770)^2}{386} = 10{,}400$ psi.

The maximum stress taking into account bending is 12,900 psi, an increase of $\dfrac{12{,}900 - 10{,}400}{10{,}400}(100) =$ 24% over that obtained with a very simple case of a free rotating ring for the particular given data.

(c) The axial force in each spoke is $F = \dfrac{2qv^2}{3gC} = \dfrac{2(27.2)(3770)^2}{3(386)(8.10)} = 82{,}500$ lb.

The stress in each spoke $= F/A_1 = 82{,}500/16 = 5160$ psi.

SUPPLEMENTARY PROBLEMS

15. Given the same data as for Problem 14 except that cast iron is to be used instead of steel, determine the maximum stress in the rim. Cast iron weighs 0.26 lb/in³. *Ans.* 11,700 psi

16. A cast iron flywheel rim is 4″ thick by 8″ wide and has mean radius 36″. The maximum tensile stress in the rim is to be limited to 8000 psi. The cross section area of each of 4 spokes is [illegible]. Cast iron weighs 0.255 lb/in^3.

(a) Determine the maximum velocity at the mean radius without exceeding the maximum [illegible] for the following:

(1) Assuming the rim is to be treated as a free rotating ring.

(2) Analyzing for the section of the rim at the spoke (ϕ = 45°).

(3) Analyzing for the section of the rim midway between spokes (ϕ = 0°).

(b) Determine the maximum stress in a spoke for the maximum velocity determined.

Ans. (a) 3480 in/sec, 2870 in/sec, 3150 in/sec. Maximum velocity without exceeding [illegible], as dictated by the most stressed section occurring in the rim where the spoke is [illegible].

(b) 1190 psi (for v = 2870 in/sec)

APPROXIMATE STRESSES IN FLYWHEEL RIMS of ordinary construction are given by the Lanza equation:

$$s = V^2\left(0.075 + \frac{0.254 d}{n t^2}\right)$$

where s = tensile stress, psi
d = mean diameter of rim, inches
t = thickness of rim, inches
V = velocity of the mean radius, ft/sec
n = number of spokes

where the effects of inertia loading and bending are accounted for by the two terms, with approximately 3/4 of the stress being due to the tensile stress in the rim due to the inertia loading and 1/4 of the stress being due to the bending of the rim considered as a beam fixed at the ends (at the spokes) and loaded between spokes by the inertia load.

SOLVED PROBLEMS

17. Assuming the maximum stress in a flywheel rim can be approximated, as suggested by Lanza, by adding 3/4 of the stress computed by considering the rim as a free rotating ring and 1/4 of the stress computed by considering the rim as a straight beam of length equal to the arc between spokes, fixed at both ends, and loaded uniformly with inertia forces, derive the equation for the maximum stress s. Take the weight density as 0.270 lb/in^3.

Solution:

(a) The stress s_1 (psi) in a free rotating ring is

$$s_1 = \frac{\rho v^2}{g} = \frac{0.270(12V)^2}{386} = 0.101V^2$$

where v = velocity in in/sec, V = velocity in ft/sec, g = 386 in/sec^2.

(b) Consider next the bending stress s_2 in the rim treated as a straight beam fixed at both ends, the length being the arc distance between spokes.

$$s_2 = \frac{Mc}{I} = \frac{6M}{bt^2} \quad \text{for a rectangular section beam}$$

The bending moment M at the ends of a beam rigidly held at both ends and uniformly loaded is

$$M = \frac{1}{12}WL = \frac{1}{12}\left[\frac{\pi d b t \rho}{n g}\left(\tfrac{1}{2}d\omega^2\right)\right]\frac{\pi d}{n} = \frac{[illegible]}{n^2}$$

where W = inertia load = (mass)(acceleration) = $\frac{(\pi d)(bt)\rho}{ng}\left(\tfrac{1}{2}d\omega^2\right)$, lb

d = mean diameter of rim, in.; b = width, in.; t = thickness, in.
ω = angular velocity, rad/sec; n = number of spokes; $L = \pi d/n$, in.

Chapter 27

Projects

The following projects are suggested for practice in applying the principles of the preceding chapters in more comprehensive situations. These projects involve, in varying degrees, combinations of analysis, synthesis, ingenuity, proportioning of parts, use of codes, drawing and sketching, selection of materials, safety considerations, economic factors, life expectancy, and other related ideas.

1. Design of a Hand Screw Press – one ton capacity

A one ton capacity hand screw press for general usage is to be designed. Preliminary specifications have been established as shown in Fig. 27-1. While a complete design would require the making of the detailed drawings, this project will involve only the analysis of the various parts together with an assembly layout drawing on 18" × 24" paper. A scale sketch of each proposed part as necessary for analysis purposes is to accompany the calculations. Sufficient information should be shown on the scale sketches to permit a draftsman to make detail drawings.

The layout drawing is to show two views: the one shown in the figure and a side view.

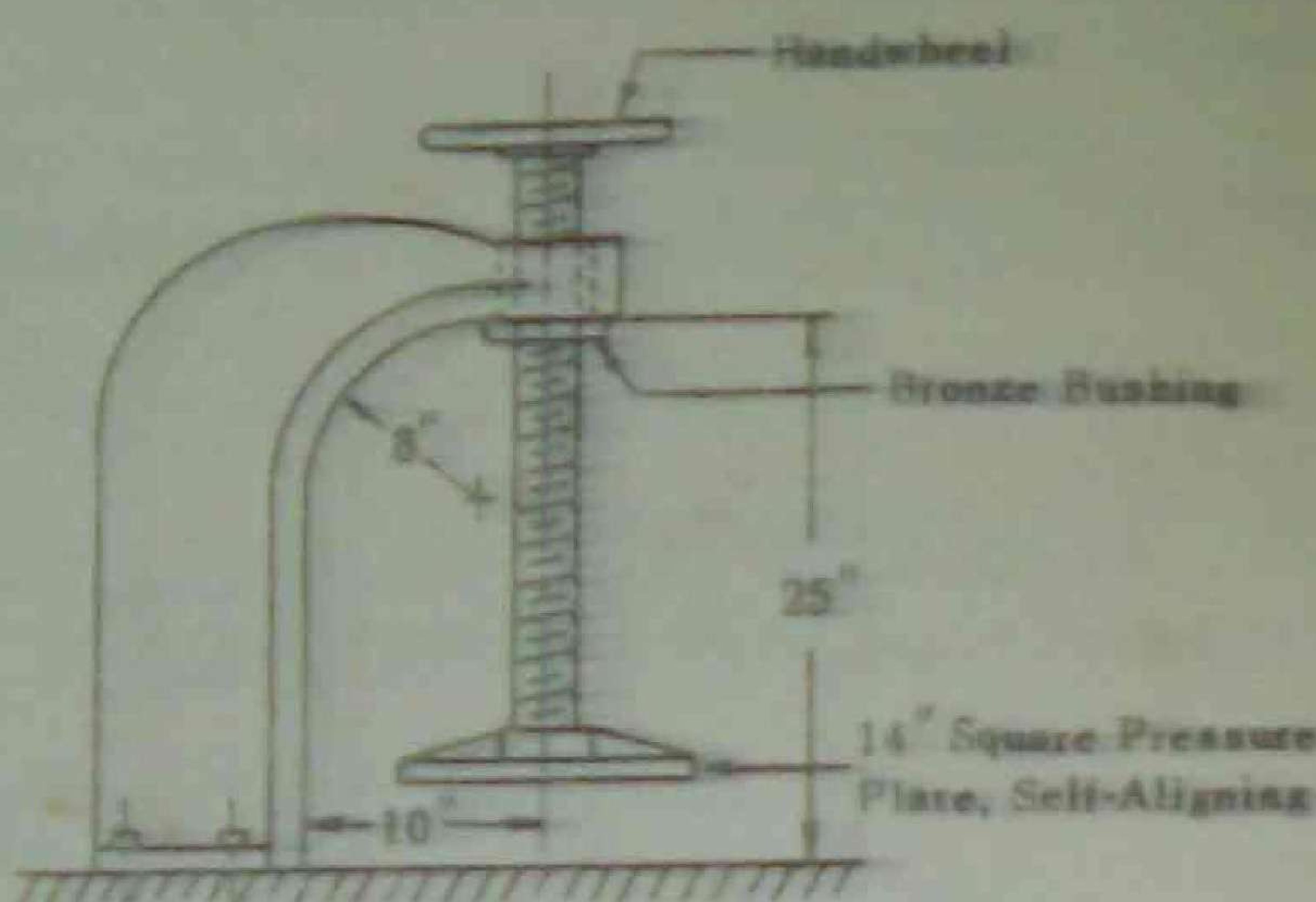

Fig. 27-1

Suggested Materials

1. Frame – Cast Iron
2. Pressure Plate – Cast Iron
3. Screw – SAE 1025 Steel
4. Bushing – Mn. Bronze
5. Bolts – Steel

The following questions indicate some of the factors to be considered in the design.

A. Design of Screw

(1) What type of thread should be used? Acme thread? Square thread? American Standard or Unified National coarse or fine thread?

(2) May this screw be treated as a short compression member? A column?

(3) If a column, what is the unsupported length? What is the end conditions constant?

(4) Is there an axial load? A bending load? A torsional load? An eccentric load? Any other load?

(5) Is the type of loading the same on either side of the nut?

(6) What combination of loading will cause the maximum stress and at what point (or points) will it occur?

(7) What should be the efficiency of the screw? Is overhauling desirable?

(8) What provisions, if any, should be made for lubrication?

(9) What factor of safety is required? On what physical properties of the materials should it be based? Ultimate strength? Endurance limit? Yield point? What is the danger of overload? To press? To operators?

6. A 3 inch diameter cast iron rod is subj... sional moment of 2500 in-lb as shown in Fig. 2-18. Determine the maximum and minimum normal stresses.

Solution:

$\sigma_y = 0$

$\sigma_x = -\dfrac{(12{,}000)(4)}{\pi 3^2} = -1700$ psi

$\tau_{xy} = \dfrac{(2500)(1.5)(32)}{\pi 3^4} = 472$ psi

$\sigma_n(\max) = -1700/2 + \sqrt{(1700/2)^2 + (472)^2}$
$= +122$ psi (tension)

$\sigma_n(\min) = -1822$ psi (compression)

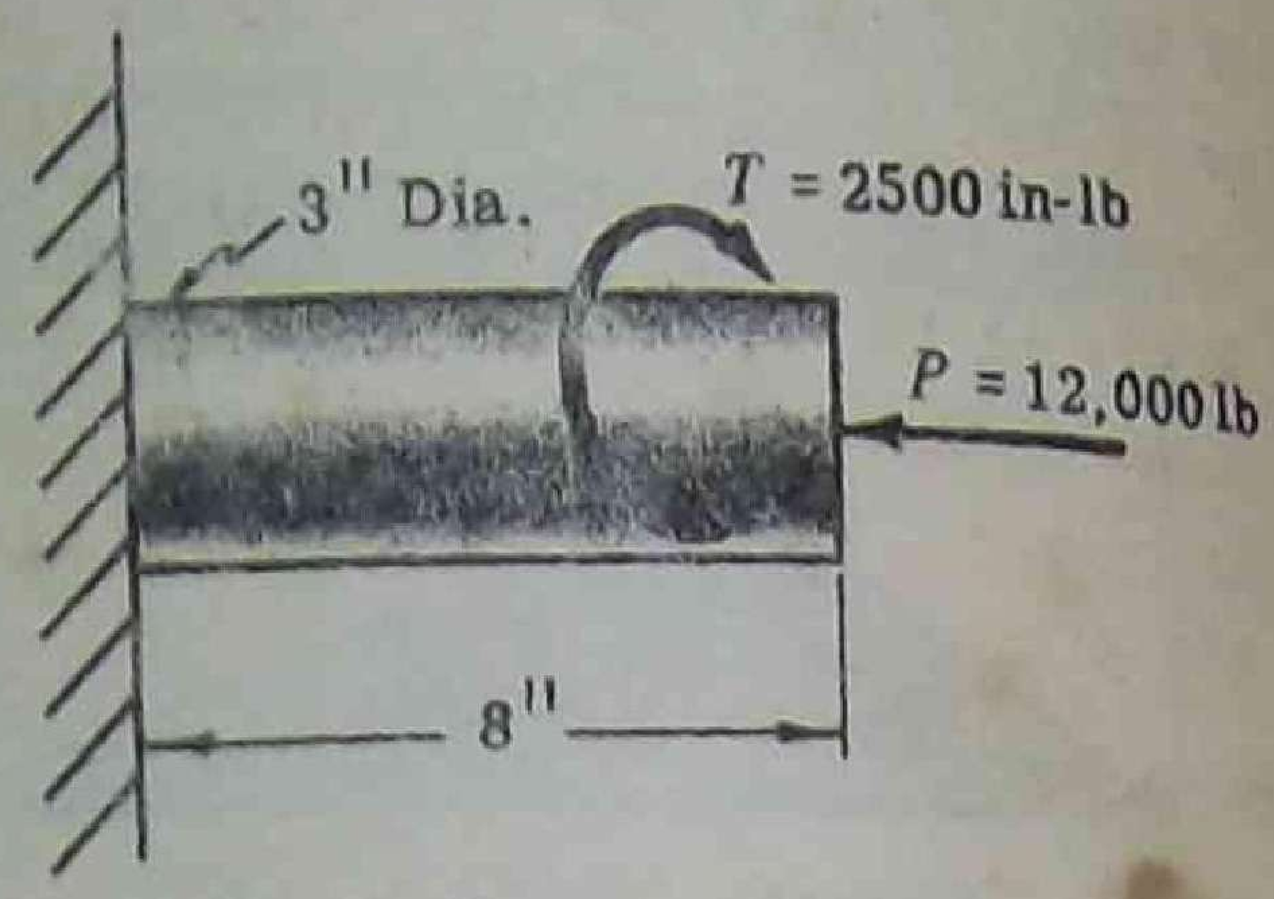

Fig. 2-18

7. Calculate the maximum numerical normal stress and the maximum shear stress at section A-A in the member loaded as shown in Fig. 2-19.

Solution:

$T = (200)(8) = 1600$ in-lb due to the 200 lb load
$M = (500)(8) = 4000$ in-lb due to the 500 lb load
$M = (200)(10) = 2000$ in-lb due to the 200 lb load

The total bending moment is the vector sum of the two bending moments.

$M(\text{total}) = \sqrt{4000^2 + 2000^2} = 4470$ in-lb

$\sigma_x = -\dfrac{P}{A} - \dfrac{Mc}{I} = -\dfrac{500}{\pi} - \dfrac{(4470)(1)(64)}{\pi 2^4} = -5849$ psi

$\tau_{xy} = \dfrac{Tr}{J} = \dfrac{16T}{\pi d^3} = \dfrac{(16)(1600)}{\pi 2^3} = 1020$ psi

$\sigma_n(\min) = -5849/2 - \sqrt{(5849/2)^2 + (1020)^2} = -6025$ psi (compression)

$\tau(\max) = \sqrt{(5849/2)^2 + (1020)^2} = 3100$ psi (shear)

Note that $\sigma_n(\min)$ is the maximum numerical normal stress.

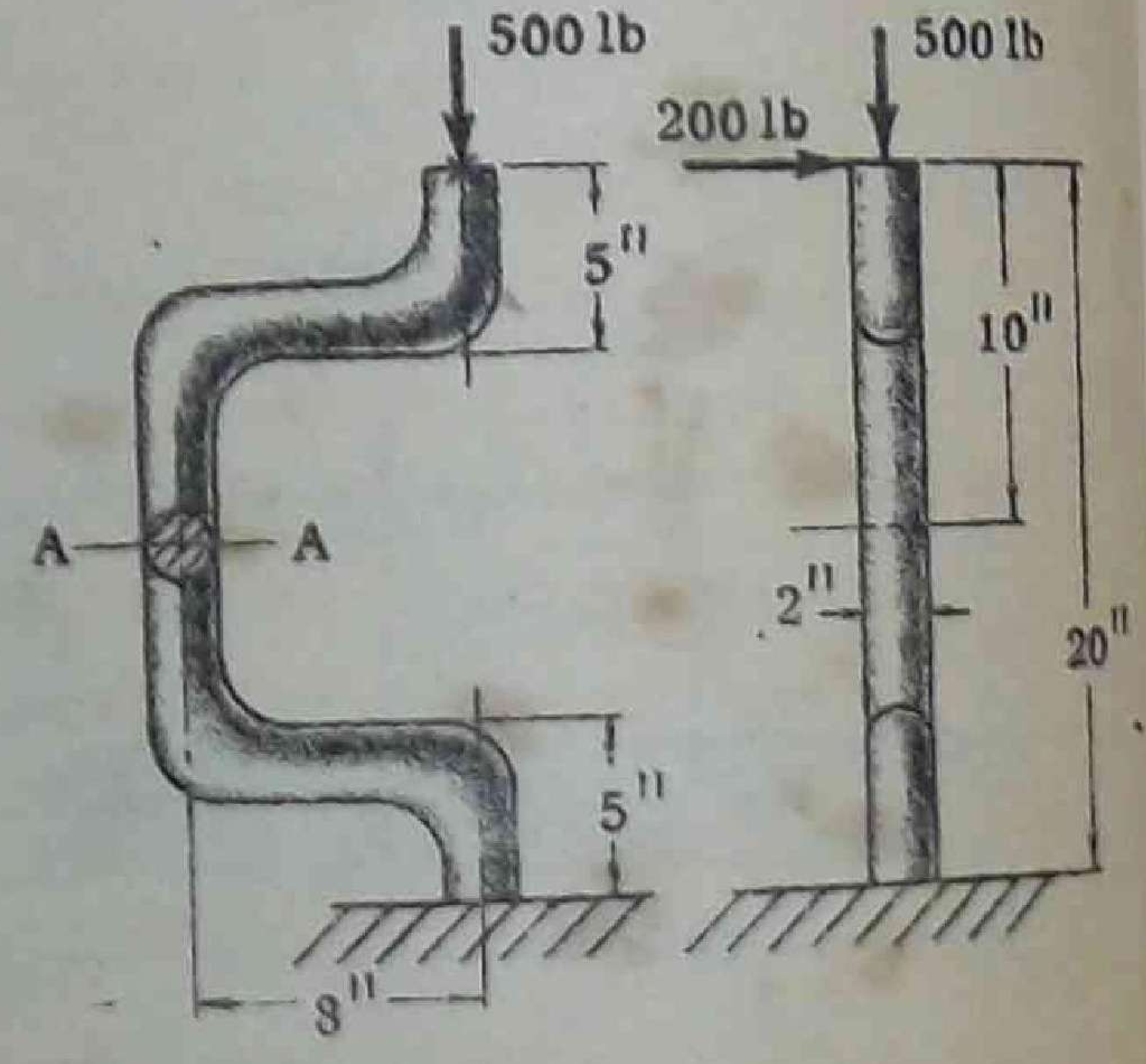

Fig. 2-19

8. Determine the required thickness of the steel bracket at section A-A, when loaded as shown in Fig. 2-20, in order to limit the tensile stress to 10,000 psi.

Solution:

$M = (1000)(2)$
$= 2000$ in-lb at section A-A

$\dfrac{P}{A} = \dfrac{1000}{2b}$

$\sigma_n(\max) = \sigma_x = \dfrac{P}{A} + \dfrac{Mc}{I}$
$= \dfrac{1000}{2b} + \dfrac{(2000)(1)(12)}{2^3 b}$
$= 10{,}000$ psi

$b = 0.35$ in. required to limit the stress to 10,000 psi.

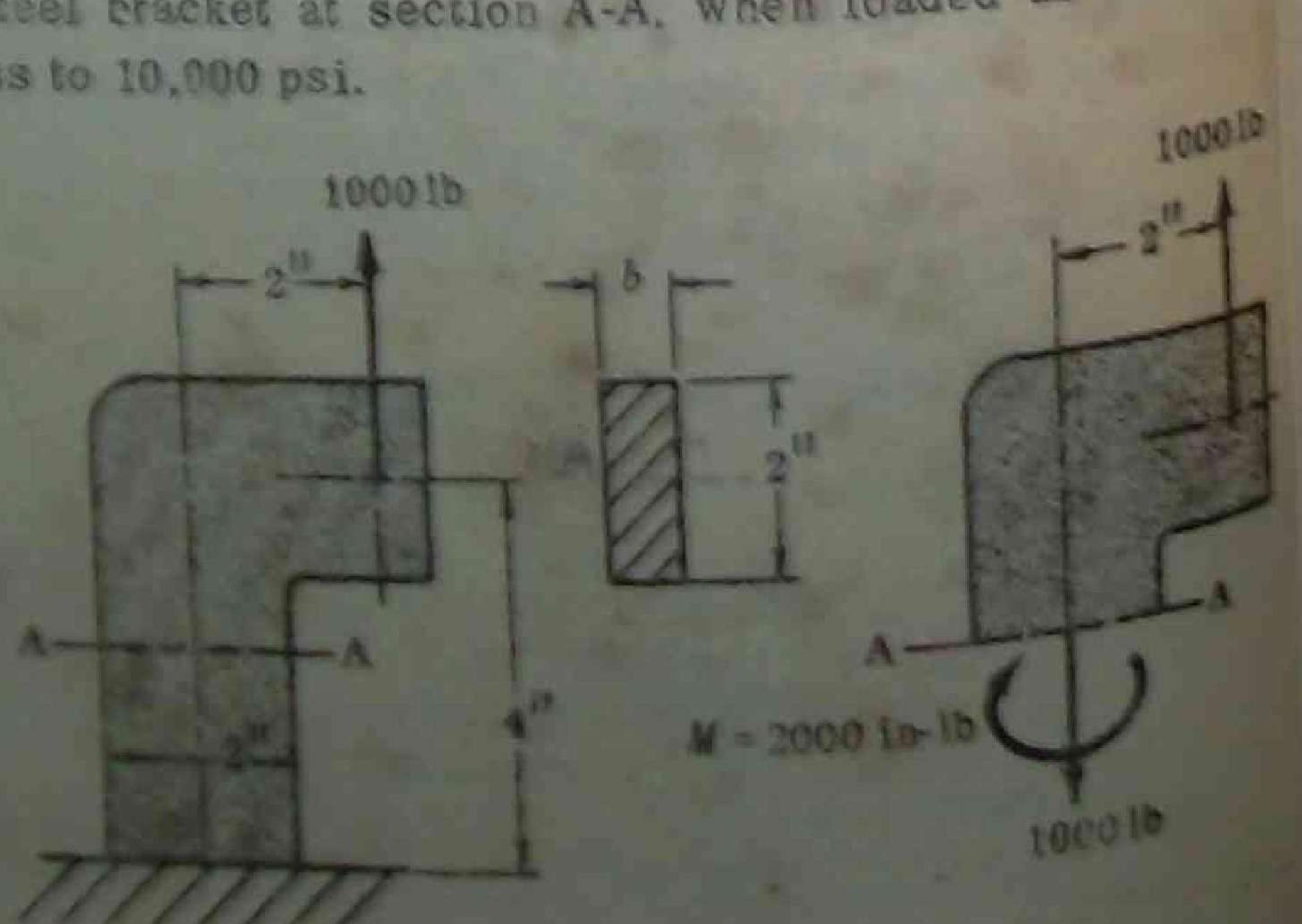

Fig. 2-20

Deflection and Buckling of Machine Members

RIGIDITY may in some cases determine the design of a machine member. The member may be strong enough to prevent a stress failure, but may not be sufficiently rigid for satisfactory operation. The following topics will discuss rigidity from the standpoint of axial deflection, torsional deflection, deflection due to bending, deflection due to shear, and buckling due to column effect.

AXIAL DEFLECTION δ due to an axial load F is based on Hooke's Law.

$$s = \left(\frac{\delta}{L}\right)(E) = \frac{F}{A}$$

from which

$$\delta = \frac{FL}{AE}$$

where

δ = axial deflection, inches
L = axial length of member before application of the axial load, inches
A = cross sectional area, in^2
E = modulus of elasticity, psi

TORSIONAL DEFLECTION θ° due to a torsional load on a solid circular section is

$$\theta^\circ = \frac{584\,TL}{GD^4}$$

For a hollow member of circular cross section, the angular deflection is

$$\theta^\circ = \frac{584\,TL}{G(D_o^4 - D_i^4)}$$

where

θ° = torsional deflection, degrees
T = torque, in-lb
D = diameter of solid member, inches
D_o = outside diameter of hollow member, inches
D_i = inside diameter of hollow member, inches
L = axial length of member between the applied and resisting torques, inches
G = modulus of rigidity, psi

For a solid rectangular member the torsional deflection is

$$\theta^{\circ} = \frac{57.3\,TL}{abc^3 G}$$

where

b = long side of rectangle, in.
c = short side of rectangle, in.
a = a factor depending upon the ratio of b/c as follows:

b/c =	1.000	1.500	1.750	2.000	2.500	3.000	4.000	6.000	8.000	10.000	∞
a =	0.141	0.196	0.214	0.229	0.249	0.263	0.281	0.299	0.307	0.313	0.333

G = modulus of rigidity, psi.
L = length of member, in.

LATERAL DEFLECTION due to bending only may be determined by solving the differential equation of the elastic curve of the neutral axis,

$$\frac{d^2 y}{dx^2} = \frac{M}{EI}$$

where

M = the bending moment, in-lb
I = the rectangular moment of inertia, in^4
E = modulus of elasticity, psi
y = deflection, in.
x = distance from end of member to the section where the deflection is to be determined, in.

A straight analytical solution of this equation by double integration is quite tedious for multiple loads and for beams that have changes in cross section. Easier solutions may usually be obtained by resorting to other methods such as: area moment method, conjugate beam method, the use of step functions, application of the theorem of Castigliano, or by graphical integration.

THE AREA MOMENT method for determining the deflection of a beam due to bending is based on the proposition that the vertical distance of any point A on the elastic curve of a beam from the tangent at any other point B on the elastic curve is equal to the moment with respect to the ordinate at A of the area of the M/EI diagram between the points A and B. (See Fig. 5-1)

$$\Delta = A_1\bar{x}_1 + A_2\bar{x}_2 + \ldots$$

where

A_1 = area of portion I of the M/EI diagram
$\bar{x}_1$ = distance from the ordinate at point A to the center of gravity of A_1
A_2 = area of portion II of the M/EI diagram
$\bar{x}_2$ = distance from the ordinate at point A to the center of gravity of A_2.

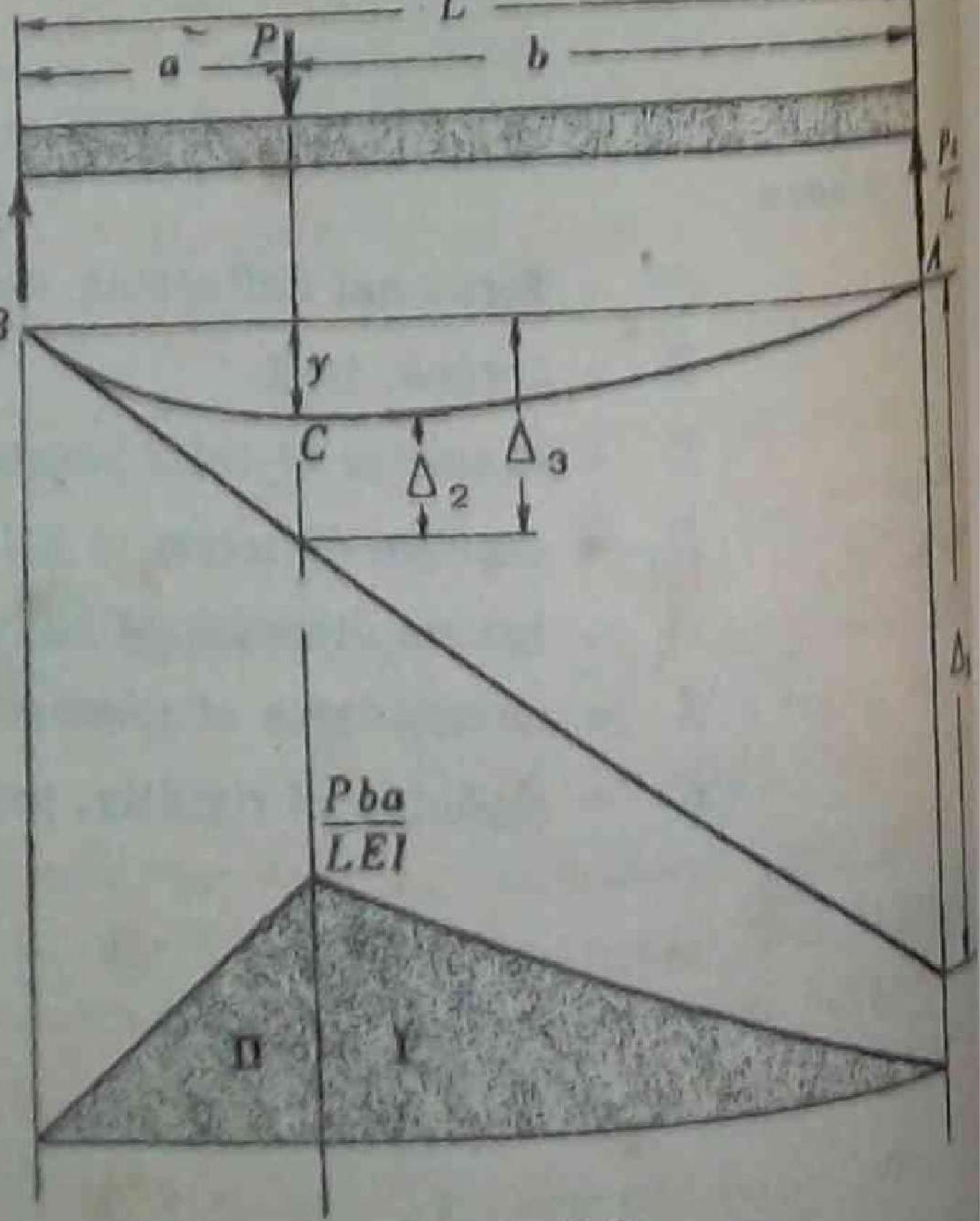

Fig. 5-1

For a solid rectangular member the torsional deflection is

$$\theta^{\circ} = \frac{57.3\,TL}{abc^3 G}$$

where

b = long side of rectangle, in.
c = short side of rectangle, in.
a = a factor depending upon the ratio of b/c as follows:

b/c =	1.000	1.500	1.750	2.000	2.500	3.000	4.000	6.000	8.000	10.000	∞
a =	0.141	0.196	0.214	0.229	0.249	0.263	0.281	0.299	0.307	0.313	0.333

G = modulus of rigidity, psi.
L = length of member, in.

LATERAL DEFLECTION due to bending only may be determined by solving the differential equation of the elastic curve of the neutral axis,

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

where

M = the bending moment, in-lb
I = the rectangular moment of inertia, in^4
E = modulus of elasticity, psi
y = deflection, in.
x = distance from end of member to the section where the deflection is to be determined, in.

A straight analytical solution of this equation by double integration is quite tedious for multiple loads and for beams that have changes in cross section. Easier solutions may usually be obtained by resorting to other methods such as: area moment method, conjugate beam method, the use of step functions, application of the theorem of Castigliano, or by graphical integration.

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$$\Delta = A_1\bar{x}_1 + A_2\bar{x}_2 + \ldots$$

where

A_1 = area of portion I of the M/EI diagram
$\bar{x}_1$ = distance from the ordinate at point A to the center of gravity of A_1
A_2 = area of portion II of the M/EI diagram
$\bar{x}_2$ = distance from the ordinate at point A to the center of gravity of A_2.

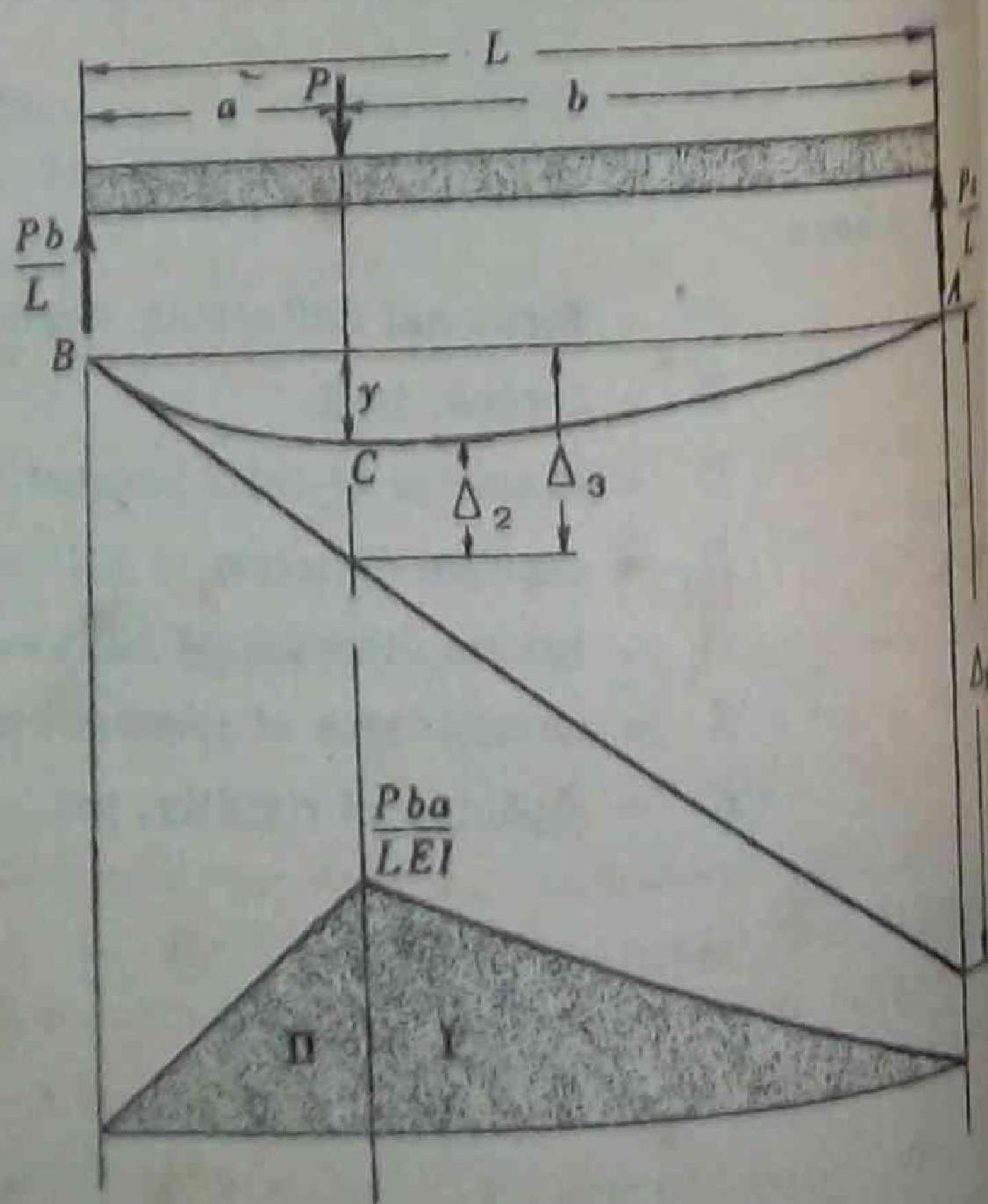


Fig. 5-1

A simple beam of length L having a concentrated load P located at a distance a from the left support and at a distance b from the right support may be used to illustrate the above procedure. Referring to Fig. 5-1, in order to determine the deflection y under the load P complete the following steps:

1. Sketch the elastic curve.
2. Sketch a tangent to the elastic curve at point B located at the left reaction.
3. Sketch the M/EI diagram.
4. Determine Δ_1 by summing the moments of the areas of section I and section II with respect to the right support:

$$\Delta_1 = \left(\frac{Pb^2a}{2LEI}\right)\left(\frac{2b}{3}\right) + \left(\frac{Pba^2}{2LEI}\right)\left(b + \frac{a}{3}\right) = \frac{Pb^3a}{3LEI} + \frac{Pb^2a^2}{2LEI} + \frac{Pba^3}{6LEI}$$

5. Determine Δ_2 which is equal to the moment of section II with respect to a vertical axis through point C:

$$\Delta_2 = \left(\frac{Pba^2}{2LEI}\right)\left(\frac{a}{3}\right) = \frac{Pba^3}{6LEI}$$

6. Determine Δ_3 by proportion:

$$\Delta_3 = \frac{a\Delta_1}{L} = \frac{Pb^3a^2}{3L^2EI} + \frac{Pb^2a^3}{2L^2EI} + \frac{Pba^4}{6L^2EI}$$

7. Then $y = \Delta_3 - \Delta_2 = \dfrac{Pb^3a^2}{3L^2EI} + \dfrac{Pb^2a^3}{2L^2EI} + \dfrac{Pba^4}{6L^2EI} - \dfrac{Pba^3}{6LEI}$.

It should be noted that in the above illustration the areas of both section I and section II are positive. If any part of the M/EI diagram is negative, then the moment of that part of the M/EI diagram must be taken as negative.

THE CONJUGATE BEAM method for determining the lateral deflection due to bending of a beam is based on the mathematical similarity of the loading, shear, and bending moment diagrams to the M/EI loading, slope, and deflection diagrams:

$$f(x) = w = \frac{dV}{dx} = \frac{d^2M}{dx^2}, \qquad f(x) = \frac{M}{EI} = \frac{d\alpha}{dx} = \frac{d^2y}{dx^2}$$

Due to the similarity of the above equations, the following statements are the basis of the solution:

1. The "shear" force of the conjugate beam is equivalent to the slope of the actual beam.
2. The "bending moment" of the conjugate beam is equivalent to the deflection of the actual beam.

It is necessary, however, to first set up the conjugate beam such that boundary conditions are satisfied. Where the slope of the original beam is not zero, a "shearing" force on the conjugate beam must exist. If the loading is such that no "shear" is present, a "shear" force must be inserted in the conjugate beam loading. Similarly with the deflection — if the deflection is not zero, a "moment" must exist. If the loading is such that no "moment" is present, a "moment" must be inserted in the conjugate beam loading.

In order to demonstrate the above procedure, consider a cantilever beam having a constant cross section and a concentrated load P at the end as shown in Fig. 5-2 below. The following steps will determine the deflection at the end of the beam:

a. Sketch the bending moment diagram.

b. Load the conjugate beam so that the intensity of load at any section is equal to the ordinate of M/EI.

c. In order to satisfy the boundary conditions, there must be a value of slope, or "shear", at the section of the conjugate beam under the load P represented by the reaction "R". In order for a deflection, or "moment", to exist at the reaction, there must be a "moment" ("M") applied at the right end of the beam.

d. The triangular distributed "load" on the conjugate beam may then be considered as equivalent to the area of this triangle, $PL^2/2EI$, concentrated at the centroid of the triangle.

e. By summation of vertical "forces", the "reaction" at the right end of the beam is $PL^2/2EI$.

f. Taking moments at the right end of the conjugate beam, we have

$$-\frac{PL^2}{2EI}\left(\frac{2L}{3}\right) + "M" = 0$$

or "M" $= PL^3/3EI$ which is the deflection at the right end of the beam.

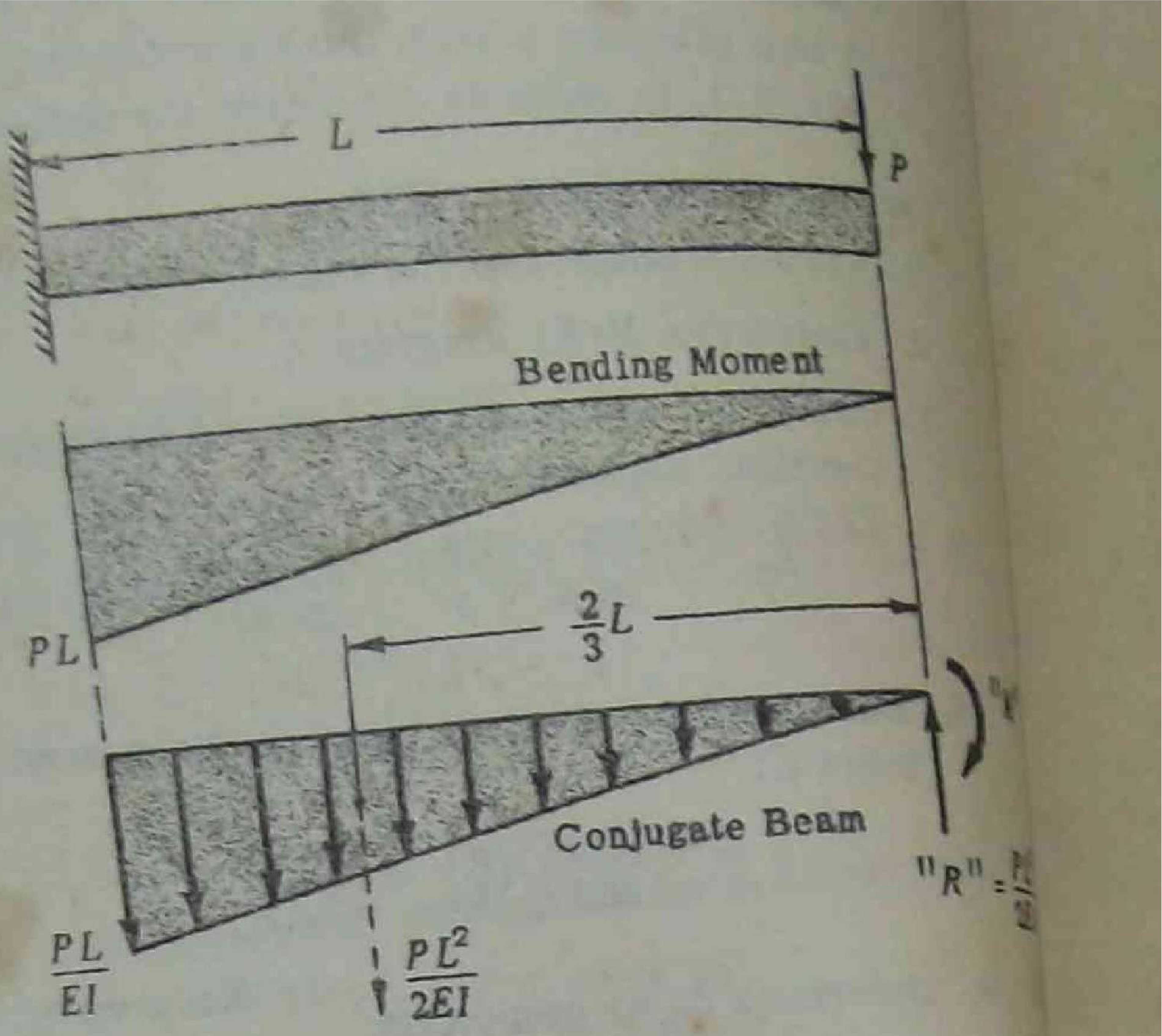


Fig. 5-2

APPLICATION OF STEP FUNCTIONS for determining the deflection of a beam due to bending requires the evaluation of only two constants of integration for a beam subjected to any number of loads and changes of section. In applying the usual double integration method of writing the M/EI equation in each section of a beam, one must evaluate two constants of integration for each section.

The use of step functions provides a means of writing mathematically a single expression for M/EI that is valid for any section of the beam, which, after double integration, results in a single expression for deflection valid for any section of the beam.

A step function, as used later, is defined by the following notation:

H_a is a step function where $H_a = 0$ if $x < a$
$H_a = 1$ if $x \geqq a$

H_b is a step function where $H_b = 0$ if $x < b$
$H_b = 1$ if $x \geqq b$

The product of the two step functions then is, for $b > a$,

$$H_a H_b = 0 \text{ if } x < b$$
$$H_a H_b = 1 \text{ if } x \geqq b$$

A plot of the above step functions is shown in Fig. 5-3.

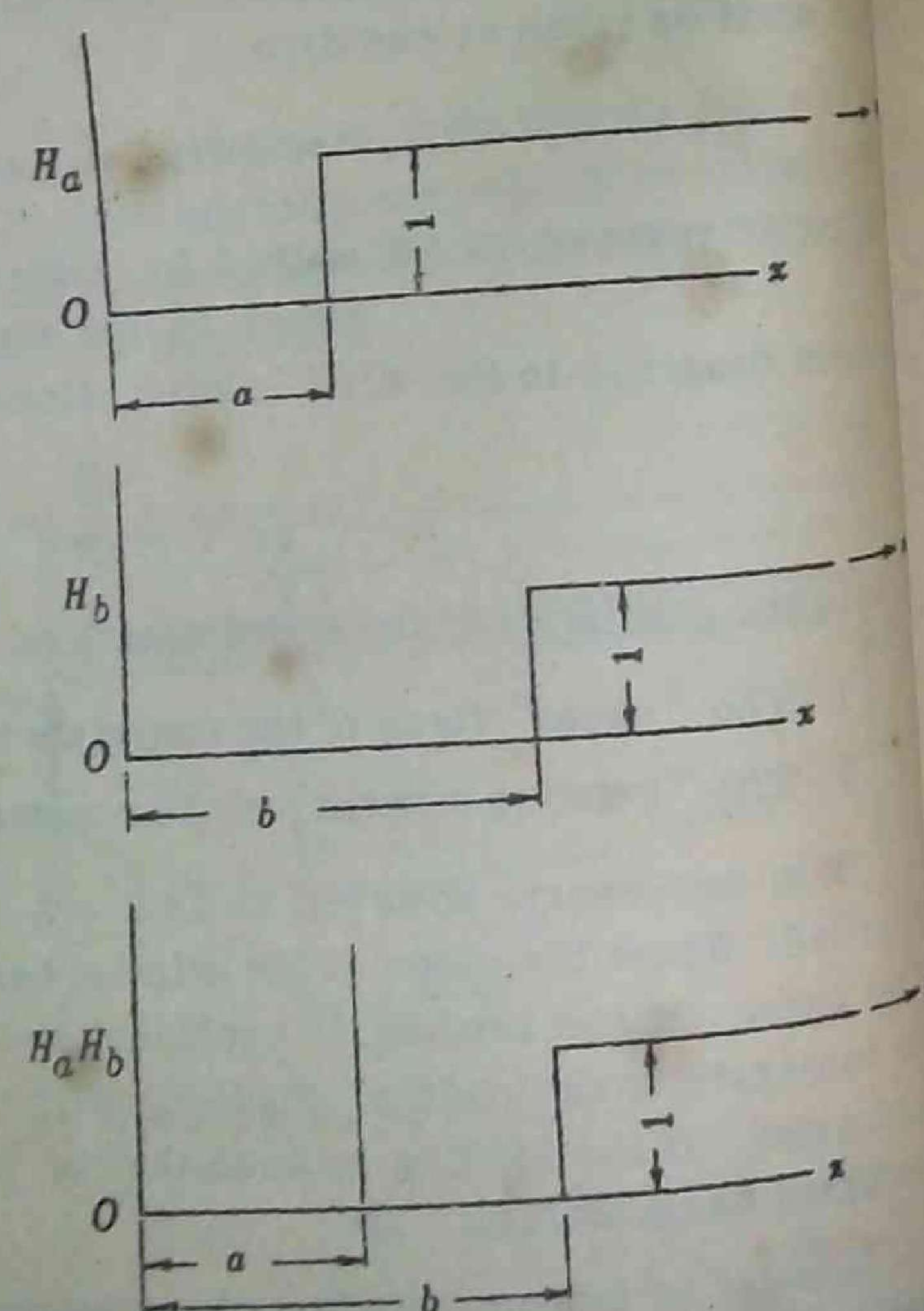


Fig. 5-3

The mathematical procedure for integrating a step function multiplied by a function $f(x)$ is

$$\int_0^x H_a f(x)\,dx = H_a \int_a^x f(x)\,dx$$

Example 1. Let $f(x) = x^2$.

$$\int_0^x H_a x^2\,dx = H_a \int_a^x x^2\,dx = \left[H_a \frac{x^3}{3}\right]_a^x + C = H_a \frac{x^3 - a^3}{3} + C$$

where C = constant of integration.

Example 2. When $b > a$,

$$\int_0^x H_b H_a (x-a)\,dx = H_b \int_b^x (x-a)\,dx$$

$$= H_b \left[\frac{(x-a)^2}{2}\right]_b^x + C = H_b \frac{(x-a)^2 - (b-a)^2}{2} + C$$

Example 3. The method of handling changes of section by means of step functions is demonstrated as follows: Fig. 5-4 shows a beam having three sections of different moments of inertia. F_1 and F_2 are overhung loads and there are fixed bearings at R_L and R_R. The moment equation valid for any section is

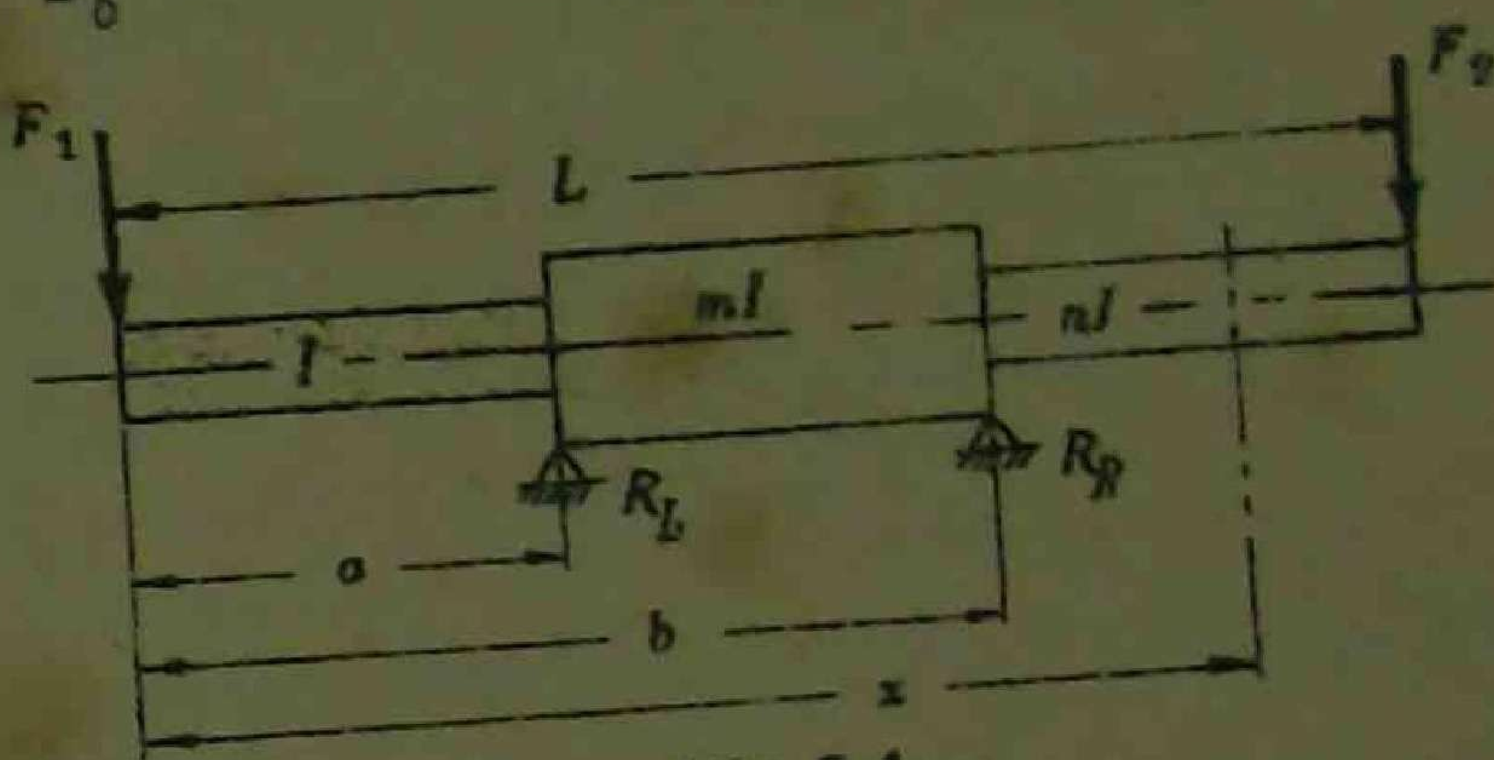


Fig. 5-4

$$M = -F_1 x + R_L (x-a) H_a + R_R (x-b) H_b$$

Now $1/I_x$, the reciprocal of the moment of inertia at any section, may be written

$$\frac{1}{I_x} = \frac{1}{I}\left[1 - H_a + \frac{H_a}{m} - \frac{H_b}{m} + \frac{H_b}{n}\right]$$

from which

$$IE\frac{d^2y}{dx^2} = \left[-F_1 x + R_L (x-a) H_a + R_R (x-b) H_b\right]\left[1 + H_a\left(\frac{1}{m} - 1\right) + H_b\left(-\frac{1}{m} + \frac{1}{n}\right)\right]$$

$$= -F_1 x + R_L (x-a) H_a + R_R (x-b) H_b - F_1 x H_a \left(\frac{1}{m} - 1\right)$$

$$+ R_L (x-a) H_a H_a \left(\frac{1}{m} - 1\right) + R_R (x-b) H_b H_a \left(\frac{1}{m} - 1\right)$$

$$-F_1 x H_b \left(-\frac{1}{m} + \frac{1}{n}\right) + R_L (x-a) H_a H_b \left(-\frac{1}{m} + \frac{1}{n}\right)$$

$$+ R_R (x-b) H_b H_b \left(-\frac{1}{m} + \frac{1}{n}\right)$$

Double integration may be completed as explained above, noting that $H_a H_a = H_a$ and $H_a H_b = H_b$.

DEFLECTION DUE TO SHEAR may be significant, for example, in machine members which are short in comparison to their depth, or for large diameter hollow members. In such cases the deflection due to shear should be added to the deflection due to bending. This could be of considerable importance when calculating deflections for determining critical speeds of rotating members. An expression for the deflection y_1 due to shear may be determined by considering a differential element taken at the centroidal axis of a member subjected to a transverse load as shown in Fig. 5-5.

$$\frac{dy_1}{dx} = \frac{s_s \text{ (at neutral axis)}}{G} + C_1 = \frac{VQ_{max}}{IbG} + C_1$$

(See Chapter 2 for discussion of $\frac{VQ}{Ib}$.)

where C_1 is a constant which accounts for the angle of rotation of the cross sections with respect to the zero deflection line. (All cross sections rotate through the same angle.) Integrating,

$$y_1 = \frac{KVx}{AG} + C_1 x + C_2 \qquad \text{where } K = \frac{QA}{Ib}$$

$K = 4/3$ for a circular cross section.

$K = 3/2$ for a rectangular cross section.

(For Fig. 5-5, $C_1 = 0$. See Fig. 5-19 for illustration of C_1.)

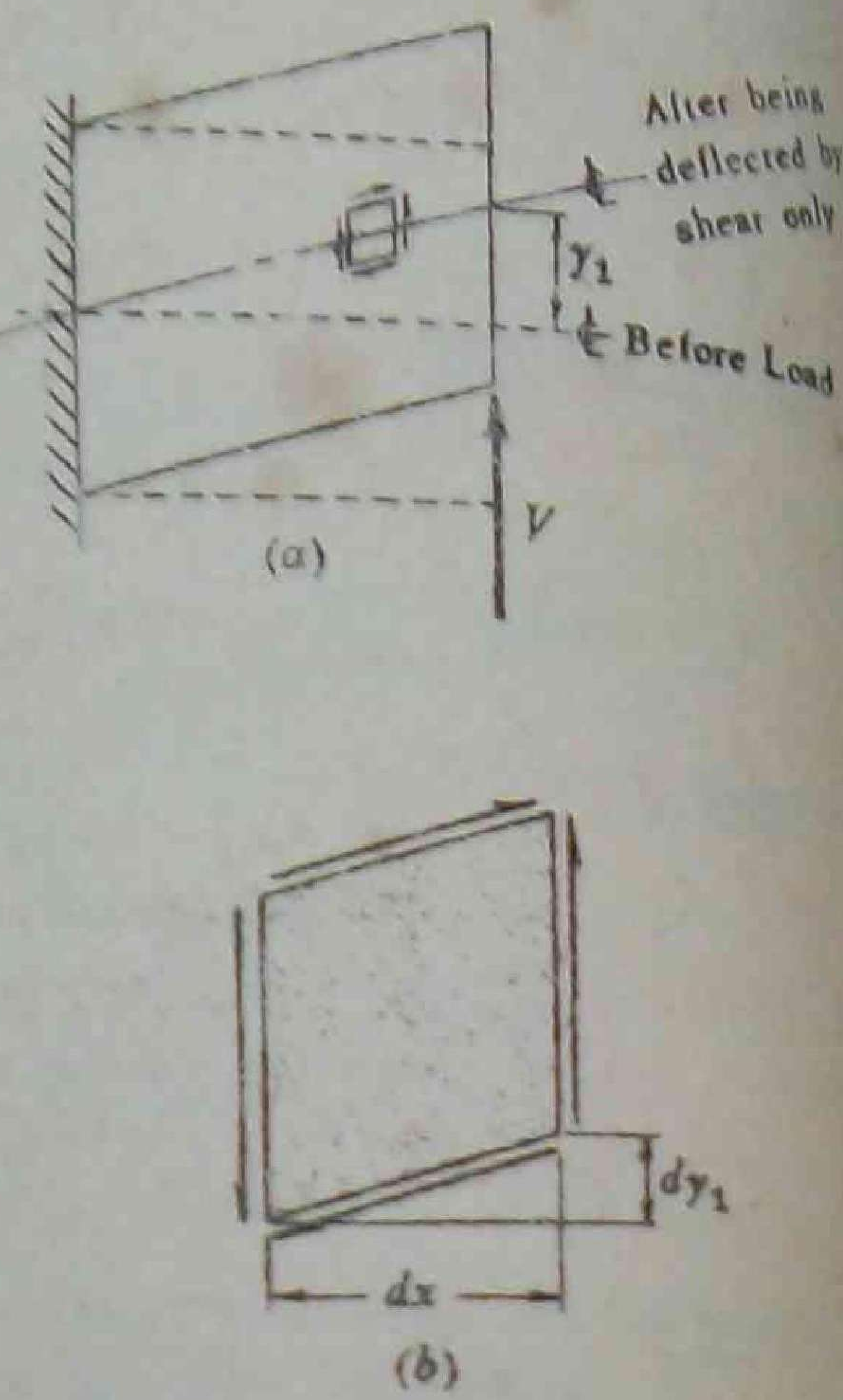


Fig. 5-5

THE THEOREM OF CASTIGLIANO may be used for determining the deflection of simple members as well as more complex structures. This theorem is based on strain energy relationships. For example, the strain energy U for a member of length L under simple tension is

$$U = \frac{F^2 L}{2AE}$$

and by taking the partial derivative of this expression with respect to F, we determine the axial deflection δ for the member in the direction of the applied force F:

$$\frac{\partial U}{\partial F} = \frac{FL}{AE} = \delta$$

Also, by taking the partial derivative of the torsional strain energy, we may determine the angle of twist for a member having a circular cross section when subjected to a torque T:

$$U = \frac{T^2 L}{2GJ} \quad \text{and} \quad \frac{\partial U}{\partial T} = \frac{TL}{GJ} = \theta \text{ (radians)}$$

In the above as well as in the following expressions, it is assumed that the material of the system follows Hooke's Law. The strain energy in a straight beam subjected to a bending moment M is

$$U = \int \frac{M^2\,dx}{2EI}$$

The strain energy in a curved beam subjected to a bending moment M is

$$U = \int \frac{M^2\,d\phi}{2AeE}$$

The strain energy in a straight member subjected to a transverse shearing force V is

$$U = \int \frac{KV^2\,dx}{2AG}$$

The strain energy in a curved beam subjected to a transverse shearing force V is

Referring to Fig. 5-6 below, the differential energy stored in the differential section due to the bending moment M, the normal force P, and the shearing force V, may be written

$$dU = \frac{M^2\,d\phi}{2AeE} + \left(\frac{P^2R\,d\phi}{2AE} - \frac{MP\,d\phi}{AE}\right) + \frac{KV^2R\,d\phi}{2AG}$$

where

$\dfrac{M^2\,d\phi}{2AeE}$ = the strain energy due to the bending moment M acting alone.

$\dfrac{P^2R\,d\phi}{2AE}$ = the strain energy due to the normal force P acting alone.

$-\dfrac{MP\,d\phi}{AE}$ = the strain energy resulting from the fact that P tends to rotate the faces of the element against the resisting couples M.

Note that in the case of Fig. 5-6, this term is negative since the force P tends to increase the angle between the two faces, while the couples M tend to decrease the angle between the two faces of the section. If the sense of P were reversed, then both P and the couples M would tend to decrease the angle between the two faces.

$\dfrac{KV^2R\,d\phi}{2AG}$ = the strain energy due to the shearing force V.

Application of the above expressions will solve deflection problems based on the theorem of Castigliano which states that the partial derivative of strain energy with respect to any force (or couple) gives the displacement (or angle of twist) corresponding to the force (or couple). In other words, if the total strain energy of a system is written as a function of one or more forces, then the deflection in the direction of any selected force may be determined by taking the partial derivative of the total strain energy with respect to the particular force selected. Also, if the total strain energy is a function of a couple as well as one or more forces, then the partial derivative of the total strain energy with respect to the couple will give the angle of rotation of the section on which the couple acts. The theorem of Castigliano may also be used to find the deflection at a point, where no load exists in the direction of the desired deflection, by adding a load Q at the selected point and in the direction in which the deflection is required. Then the partial derivative $\partial U/\partial Q$ will give the desired deflection when Q is made equal to zero.

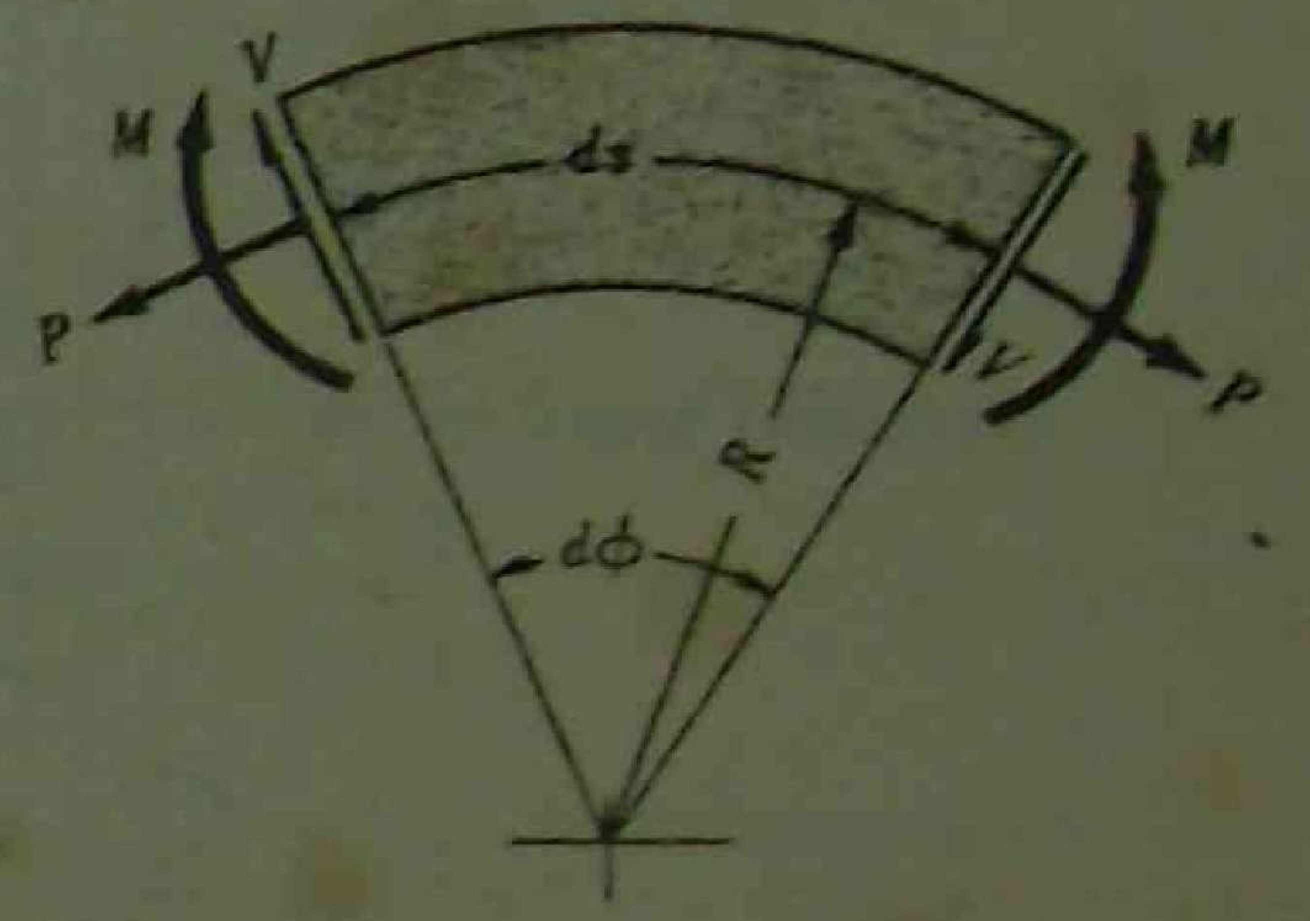


Fig. 5-6

GRAPHICAL INTEGRATION is another method to obtain a deflection curve for a shaft subjected to bending loads. The method is illustrated by the following simplified example with the following steps. Refer to Fig. 5-7.

1. Divide the area into sections, with ordinates y_1, y_2, etc., at the midpoints of x_1, x_2, etc., to locate points 1, 2, etc. (x_1 need not be equal to x_2, although these are usually taken equal for simplification in drafting).
2. Project from points 1, 2, etc., to locate points 1′, 2′, etc., on any vertical line AB. From any point O' on the horizontal axis (thus determining distance H), draw rays O'-1′, O'-2′, etc.

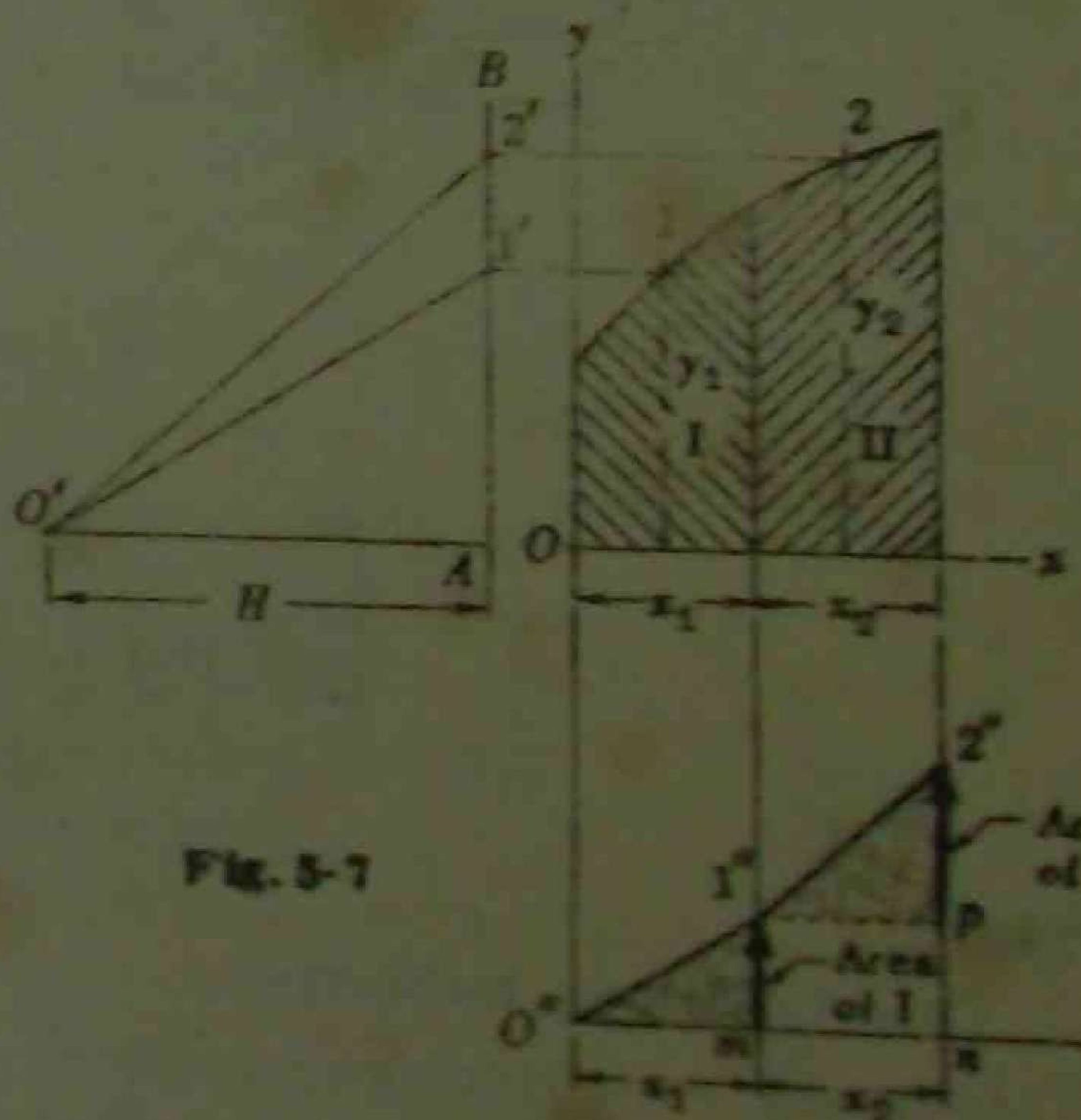


Fig. 5-7

3. Draw line O''-$1''$ parallel to line O'-$1'$ and line $1''$-$2''$ parallel to O'-$2'$. Line m-$1''$ is proportional to area I and line p-$2''$ is proportional to area II, or, line n-$2''$ is proportional to the sum of the areas I and II.

4. The proof is obtained from the property of similar triangles. Consider triangles O'-A-$1'$ and O''-m-$1''$:

$$\frac{A-1'}{O'-A} = \frac{m-1''}{O''-m} \quad \text{or} \quad \frac{y_1}{H} = \frac{m-1''}{x_1} \quad \text{or} \quad m-1'' = \frac{x_1 y_1}{H}$$

or the area $x_1 y_1 = H(m-1'')$. Thus the vertical distance $m-1''$ is proportional to the area of I which is approximated by $(x_1 y_1)$. If the distance x_1 *is* small, the approximation is very close to the true area. Thus the smaller the divisions, the closer the approximation is to the true area.

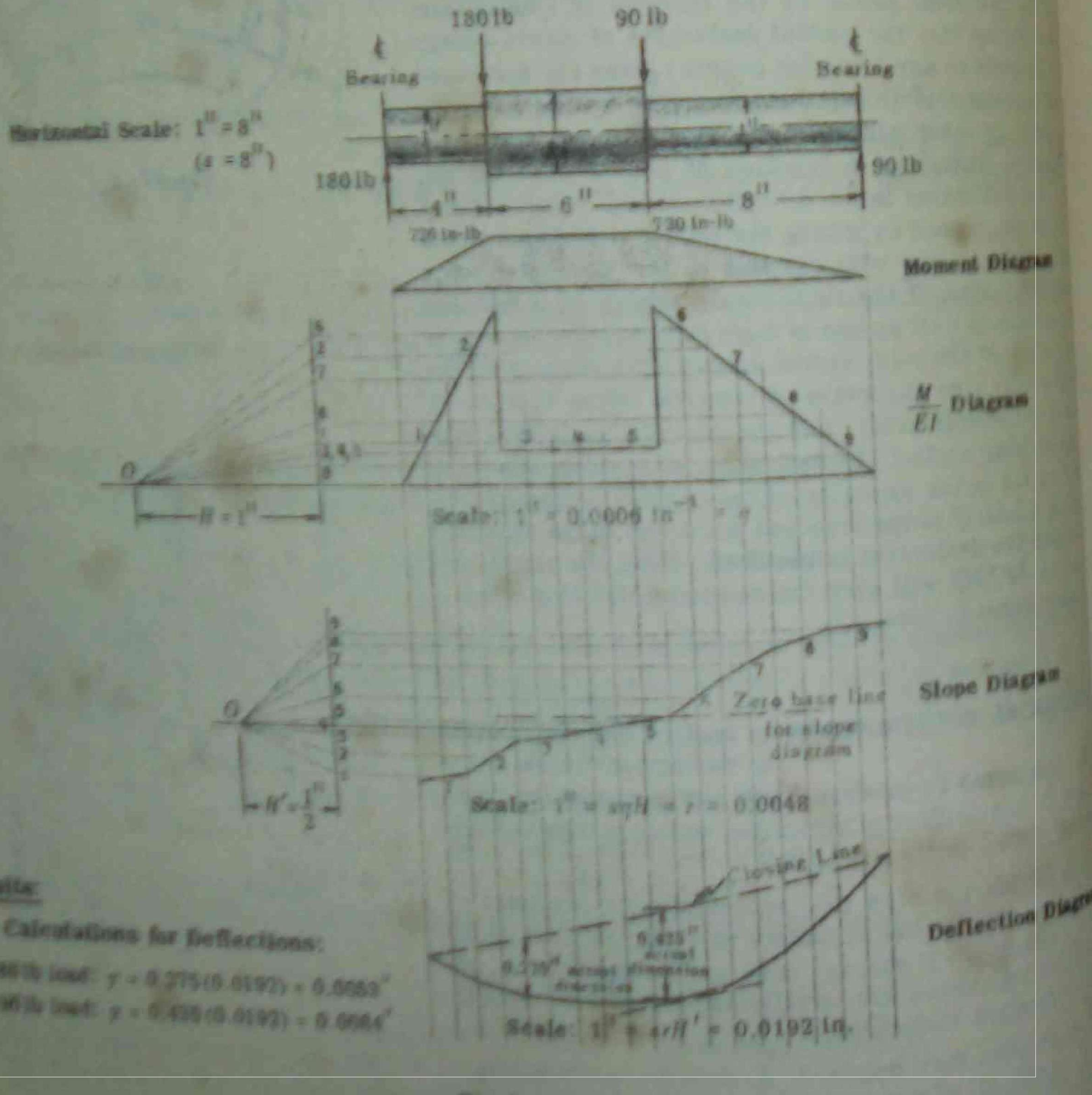
5. In a similar manner,

$$p-2'' = \frac{x_2 y_2}{H}$$

or the area $x_2 y_2 = H(p-2'')$. Then the total intercept $n-2''$ is the sum of the two areas shown.

6. The above procedure is illustrated for two cases:
   (*a*) Beam supported at the ends, Fig. 5-8(*a*) below, diameters given.
   (*b*) Beam with an overhung load, Fig. 5-8(*b*) below, diameters not given.

<u>Example (*a*).</u> Determine the deflection at each load. Use double integration, graphically.



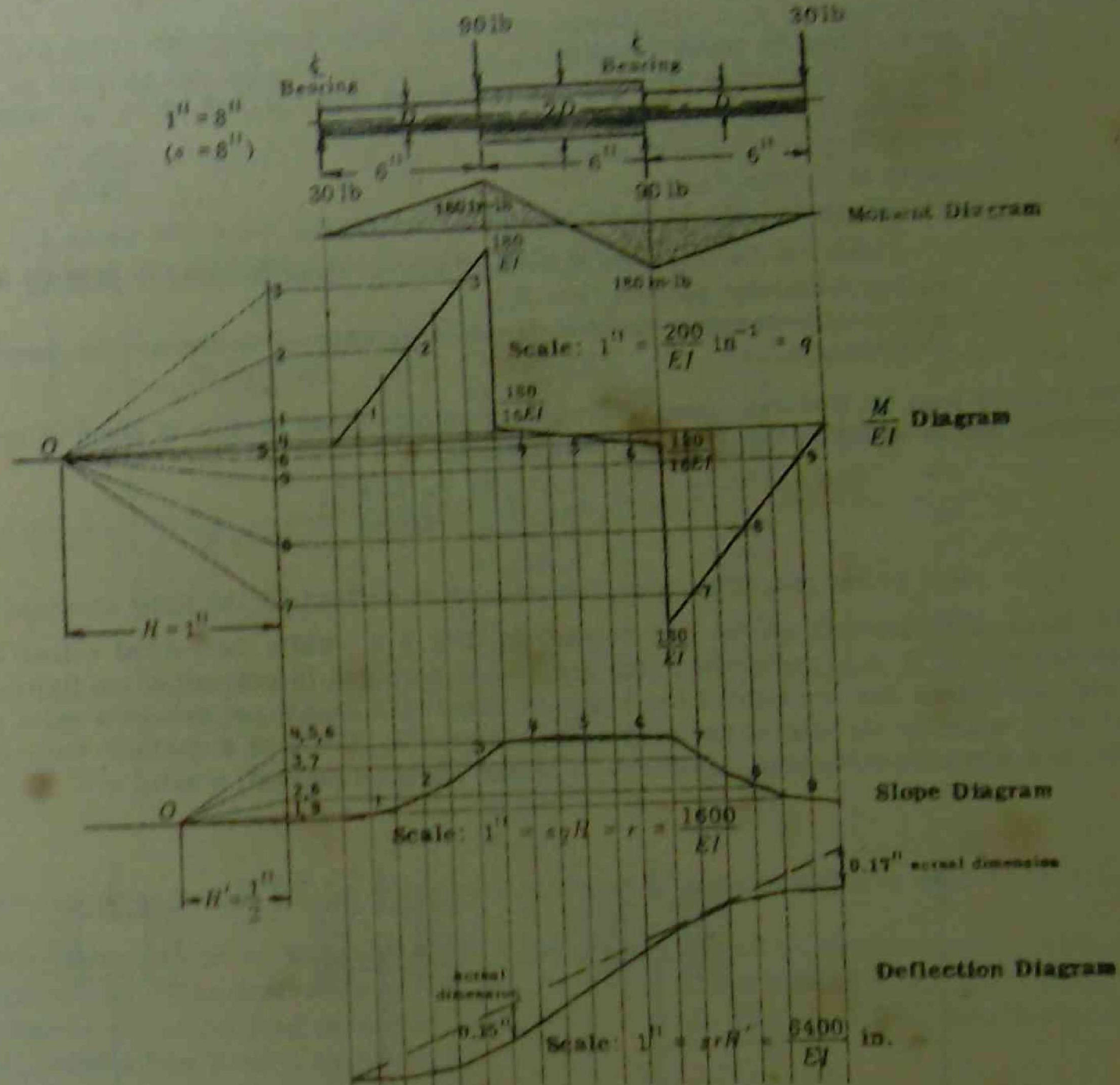
<u>Results:</u>

Calculations for Deflections:

At 180 lb load: $y = 0.275(0.0192) = 0.0053''$
At 90 lb load: $y = 0.425(0.0192) = 0.0084''$

Fig. 5-8(*a*)

Example (b). Determine diameter D to limit the deflection at the 90 lb load to 0.001". Use double integration, graphically. The moment of inertia of the sections of diameter D is I.



Result:

Deflection at 90 lb load is $y = (0.15)\left(\frac{6400}{EI}\right)$.

If the deflection at the 90 lb load is limited to 0.001",

$I = 0.032\ \text{in}^4$ for $E = 30 \times 10^6$ psi, or $D = 0.90"$.

Fig. 5-8(b)

9. The parallel side rod of a locomotive weighs 60 lb per ft. The crank length OP is 15 inches and the radius of the driver is 3 feet. If the speed of the engine is 60 mph and the tractive effort per wheel is 10,000 lb, find the maximum normal and the maximum shear stresses in the side rod due to inertia and axial loading for the position shown in Fig. 2-21. Take into account the weight of the rod. The cross section of the side rod is 3" × 6".

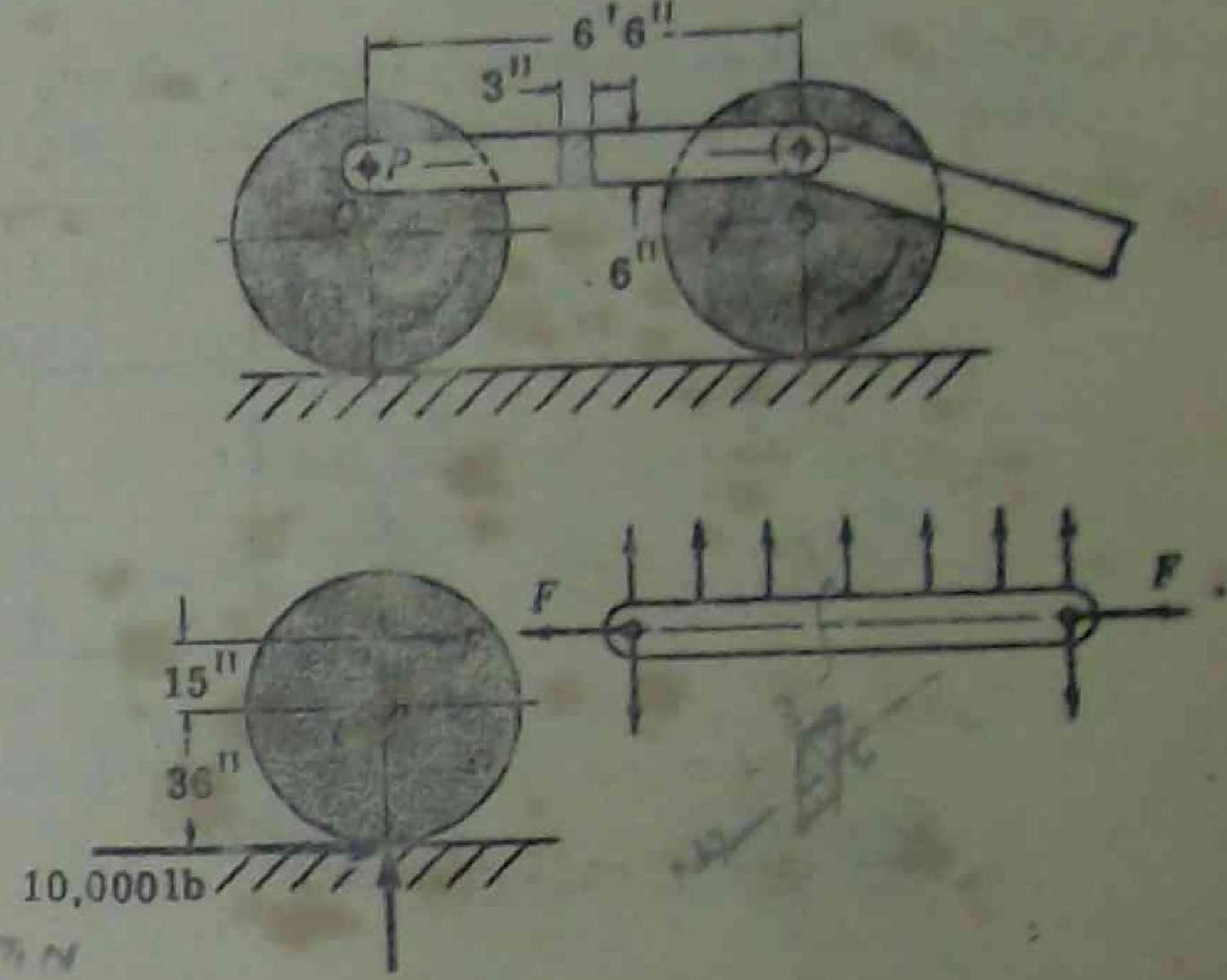


Fig. 2-21

Solution:

At 60 mph the wheels are making 4.67 rps.

All points on the side rod have a downward acceleration, a_p.

$a_p = a_o \leftrightarrow a_{po} = a_{po}$, since $a_o = 0$.

$a_{po} = r\omega^2 = (15/12)(2\pi \times 4.67)^2 = 1080 \text{ fps}^2$

Total weight of side rod = (60)(6.5) = 390 lb.

Inertia force acting upward on rod = (390/32.2)(1080) = 13,100 lb.

Net upward force on rod = 13,100 − 390 = 12,710 lb.

The axial force F can be determined by using the rear wheel and the rod as free bodies and taking the summation of moments about the center of the wheel, O.

$15F = (10{,}000)(36)$, $\quad F = 24{,}000$ lb axial load

The maximum bending moment for a simple beam carrying a uniformly distributed load is

$$WL/8 = (12{,}710)(78)/8 = 124{,}000 \text{ in-lb}$$

$$s_x = \frac{P}{A} + \frac{Mc}{I} = \frac{24{,}000}{18} + \frac{(124{,}000)(3)(12)}{(3)(6)^3} = 8230 \text{ psi}$$

$s_n(\max) = s_x = 8230$ psi (tension)

$\tau(\max) = 8230/2 = 4115$ psi (shear)

10. A Z-bracket is supported and loaded as shown in Fig. 2-22. Compute the maximum shear stress at section A-A and at section B-B.

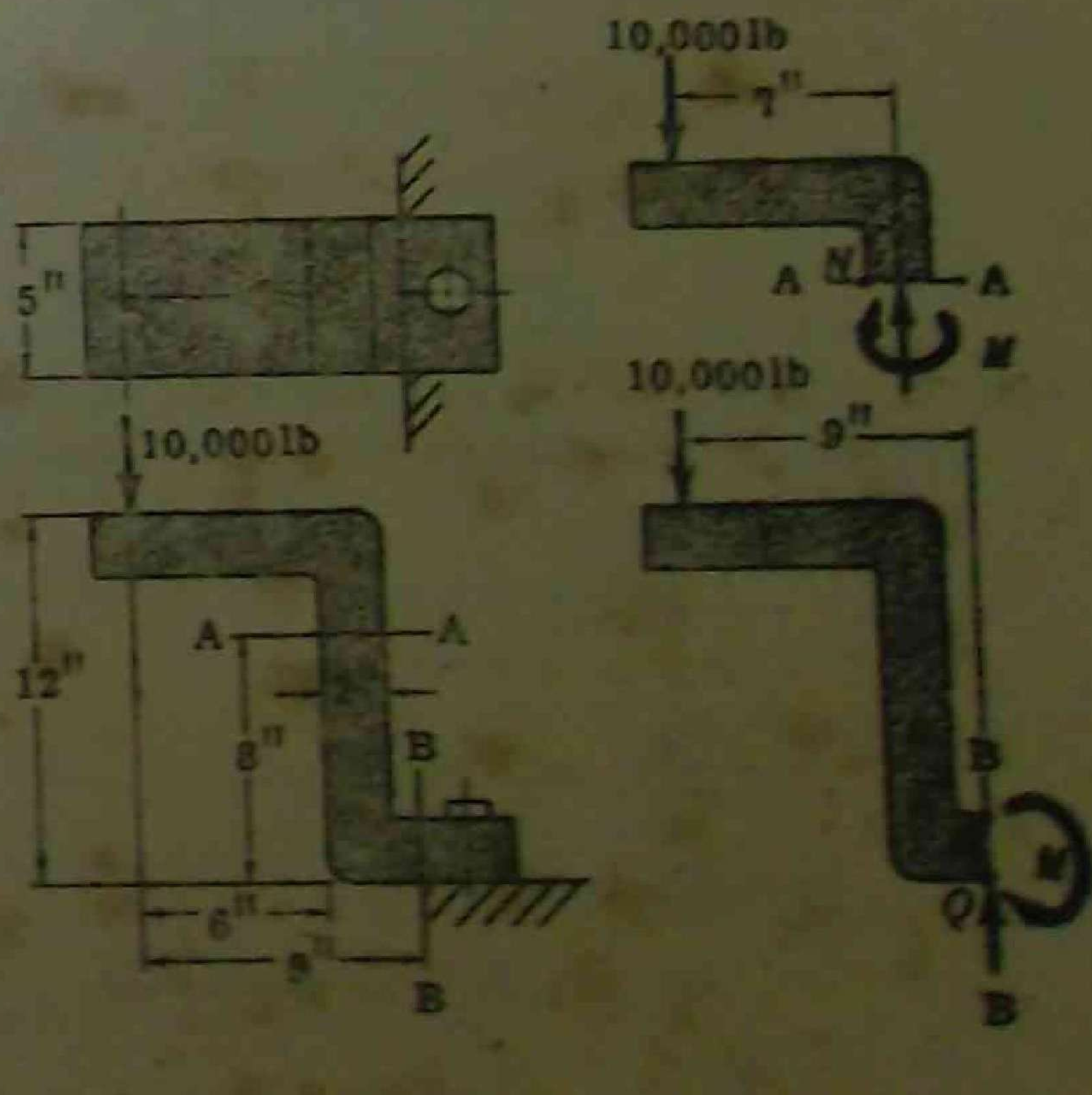


Fig. 2-22

Solution:

Using the portion of the bracket above section A-A as a free body: at point N, $s_y = 0$ and $\tau_{xy} = 0$.

$$s_x = \frac{P}{A} + \frac{Mc}{I} = -\frac{10{,}000}{10} - \frac{(10{,}000)(7)(1)(12)}{(5)(2)^3}$$

$$= -22{,}000 \text{ psi (compression)}$$

$\tau(\max) = 22{,}000/2 = 11{,}000$ psi (shear)

Using the portion of the bracket to the left of section B-B as a free body: at points Q and R, $s_y = 0$ and $\tau_{xy} = 0$.

$$s_x = \frac{Mc}{I} = \frac{(10{,}000)(9)(1)(12)}{(5)(2)^3}$$

= 27,000 psi (tension at point R and compression at point Q)

$\tau(\max) = 27{,}000/2 = 13{,}500$ psi (shear at section B-B).

COLUMN ACTION due to axial loading of machine parts occurs very frequently. If the axial load is a tensile load, then the application of $S = P/A$ is in order. If the axial load is a compressive load, then an appropriate column equation should be used.

The Euler equation for the critical load for slender columns of uniform cross section is

$$F_{cr} = \frac{C\pi^2 EA}{(L/k)^2}$$

where

F_{cr} = critical load to cause buckling.
C = constant depending upon the end conditions (see Fig. 5-9 below for values).
E = modulus of elasticity, psi.
A = area of transverse section, in.^{2}
L = length of column, in.
k = minimum radius of gyration which is $\sqrt{I/A}$ inches,

where I is the minimum moment of inertia about the axis of bending.
For a circular section, $k = D/4$.
For a rectangular section $k = h\sqrt{3}/6$, where h is the smaller dimension of the rectangle.

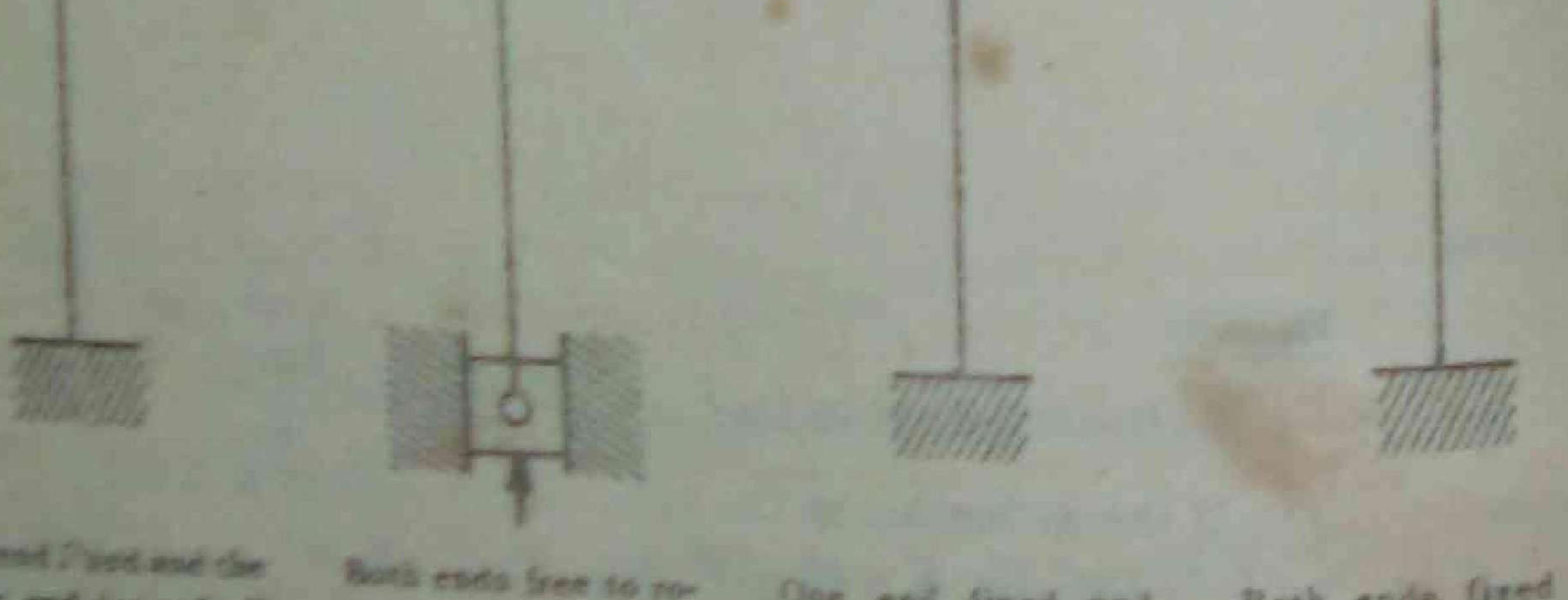
The critical load for moderate length columns of uniform cross section is given by several empirical formulas, one of which is the J. B. Johnson formula:

$$F_{cr} = s_y A\left(1 - \frac{s_y (L/k)^2}{4C\pi^2 E}\right)$$

where

s_y = yield point, psi. The other symbols are as defined for the Euler equation.

The value of C depends on the end conditions, Fig. 5-9. While theoretical values of C greater than one are given, it is recommended that great care be taken in evaluating the fixity of the ends. Where end conditions are uncertain, the value of C perhaps should not exceed a value of around ... for even what might be thought of as fixed ends. In general, $C = 1$ as a maximum value might be appropriate, and where considerable flexibility of an end might be present, a value of $C = \frac{1}{4}$ might be desirable.



One end fixed and the other end free of all restraint
$C = \frac{1}{4}$

Both ends free to rotate, but not free to move laterally (so-called round, or pinned, or hinged end columns)
$C = 1$

One end fixed and one end free to rotate, but not free to move laterally
$C = 2$

Both ends fixed so that the tangent to the elastic curve at each end is parallel to the original axis of the column
$C = 4$

The safe load is obtained by dividing the critical load by a factor of safety N:

Safe load F, Euler equation: $$F = \frac{F_{cr}}{N} = \frac{C\pi^2 EA}{N(L/k)^2}$$

Safe load F, J. B. Johnson equation: $$F = \frac{s_y A}{N}\left(1 - \frac{s_y (L/k)^2}{4C\pi^2 E}\right)$$

The value of L/k which determines whether the Euler equation or J. B. Johnson equation should be used is found by equating the critical load from the Euler equation to the critical load from the J. B. Johnson formula:

$$\frac{C\pi^2 EA}{(L/k)^2} = s_y A\left(1 - \frac{s_y (L/k)^2}{4C\pi^2 E}\right) \qquad \text{from which} \qquad L/k = \sqrt{\frac{2C\pi^2 E}{s_y}}$$

The value of L/k above which the Euler equation should be used and below which the J. B. Johnson formula should be used, for different representative data are: For theory

C	E	s_y	$(L/k)^2$	(L/k)
$\frac{1}{4}$	30×10^6 psi	80,000 psi	1,849	43
		70,000	2,113	46
		60,000	2,465	50
		50,000	2,958	54
		40,000	3,697	61
1	30×10^6 psi	80,000 psi	7,394	86
		70,000	8,451	92
		60,000	9,860	99
		50,000	11,832	109
		40,000	14,789	121
2	30×10^6 psi	80,000 psi	14,789	121
		70,000	16,902	130
		60,000	19,719	140
		50,000	23,663	154
		40,000	29,579	172

If L/k is below that given by $\sqrt{2C\pi^2 E/s_y}$, use the J. B. Johnson formula, which is valid down to $L/k = 0$.

Equivalent Column Stresses are used where column action is to be combined with other effects as torsion and bending. The equivalent stress is a fictitious stress related to the yield point stress in the same way as the actual load is related to the critical load. The equivalent column stress for an actual load F, derived from Euler's equation, is

$$s_{eq} = \frac{F}{A}\left(\frac{s_y (L/k)^2}{C\pi^2 E}\right) = \frac{F}{A}\alpha \qquad \text{where} \quad \alpha = \frac{s_y (L/k)^2}{C\pi^2 E}$$

Note that the equivalent stress depends upon the yield point stress, whereas the critical load is independent of the yield point. If a column is of given proportions and length, changing materials does not change the critical load, whereas the equivalent stress changes. The ratio of the actual load to the critical load is the same, however, as the ratio of the equivalent stress to the yield point stress.

The equivalent column stress for an actual load F, derived from J. B. Johnson's equation, is

$$s_{eq} = \frac{F}{A}\left(\frac{1}{1 - \dfrac{s_y (L/k)^2}{4C\pi^2 E}}\right) = \frac{F}{A}\alpha \qquad \text{where} \quad \alpha = \frac{1}{1 - \dfrac{s_y (L/k)^2}{4C\pi^2 E}}$$

In the equivalent stress equations, the following relations hold, with the symbols as defined above:

$$\frac{F_{cr}}{F} = \frac{s_y}{s_{eq}} = N$$

SOLVED PROBLEMS

1. Derive an expression for the elastic strain energy of bending in a straight beam.

Solution:

The change of angle $d\alpha$ between two transverse planes at a distance dx apart in a straight beam subjected to a bending moment M is $d\alpha = M\,dx/EI$. The strain energy in the section of the beam of length dx is

$$dU = \frac{M\,d\alpha}{2} = \frac{M^2\,dx}{2EI} \qquad \text{or} \qquad U = \int \frac{M^2\,dx}{2EI}$$

2. Derive an expression for the elastic strain energy of bending in a curved beam.

Solution:

The change of angle $d\alpha$ between two transverse planes separated by an angle $d\phi$ in a curved beam subjected to a bending moment M is $d\alpha = M\,d\phi/AeE$, where e is the distance from the centroidal axis to the neutral axis, always measured from the centroidal axis toward the center of curvature. The strain energy in the section between the two planes is

$$dU = \frac{M\,d\alpha}{2} = \frac{M^2\,d\phi}{2AeE} \qquad \text{or} \qquad U = \int \frac{M^2\,d\phi}{2AeE}$$

3. Determine that the deflection due to bending at the load P located at the end of a cantilever of length L is $PL^3/3EI$ by using
(a) double integration, (b) area moment method, (c) the theorem of Castigliano.

Solution:

(a) Using double integration.

The equation of the elastic curve is

$$EI\frac{d^2y}{dx^2} = M = -Px$$

where x is measured from the load toward the support. Integrating once, we have

$$EI\frac{dy}{dx} = \frac{-Px^2}{2} + C_1$$

Since the slope is zero at $x = L$, $C_1 = PL^2/2$ and

$$EI\frac{dy}{dx} = \frac{-Px^2}{2} + \frac{PL^2}{2}$$

The second integration gives

$$EIy = \frac{-Px^3}{6} + \frac{PL^2x}{2} + C_2$$

Since $y = 0$ at $x = L$, $C_2 = -PL^3/3$ and

$$EIy = \frac{-Px^3}{6} + \frac{PL^2x}{2} - \frac{PL^3}{3}$$

and for $x = 0$,

(*b*) Using area moment method, Fig. 5-10.

(*1*) Sketch the beam showing load P.

(*2*) Sketch the elastic curve and draw a tangent to the curve at point B.

(*3*) Sketch the M/EI diagram.

(*4*) The deflection y is then obtained by taking the moment of the area of the M/EI between points A and B with respect to point A. The moment arm of this area is $2L/3$. The area is $-PL^2/2EI$. Then

$$y = \left(\frac{2L}{3}\right)\left(\frac{-PL^2}{2EI}\right) = \frac{-PL^3}{3EI}$$

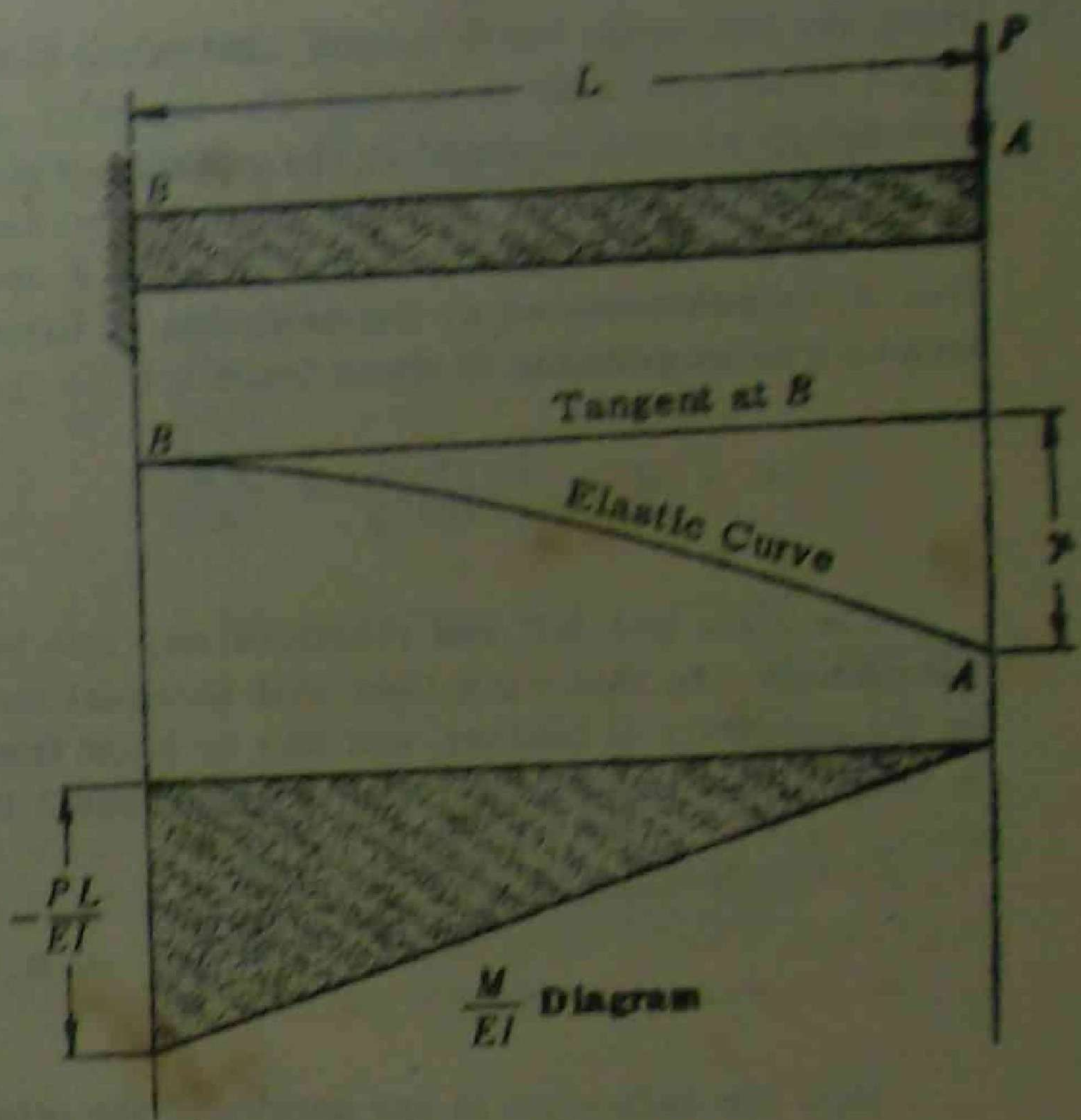


Fig. 5-10

(*c*) Using the Castigliano theorem.

The strain energy stored in a straight beam subjected to a bending moment M is

$$U = \int \frac{M^2\,dx}{2EI}.$$

Since the moment is $-Px$,

$$U = \int_0^L \frac{(-Px)^2\,dx}{2EI} = \frac{P^2L^3}{6EI}.$$

Then the deflection is obtained by taking the partial derivative $\frac{\partial U}{\partial P}$: $\quad y = \frac{\partial U}{\partial P} = \frac{2PL^3}{6EI} = \frac{PL^3}{3EI}$

A plus value of deflection means that the deflection is in the direction of the load, which is downward for this problem.

4. Determine that the deflection at the load due to bending of a simply supported beam having a concentrated load of P at its mid-length is $PL^3/48EI$ (*a*) by using the area moment method, (*b*) by using the conjugate beam method, (*c*) by using step functions, (*d*) by using the theorem of Castigliano.

Solution:

(*a*) As shown in Fig. 5-11, sketch the elastic curve, and the M/EI diagram. Then draw a tangent to the elastic curve at its midpoint A, which in this case is the point of maximum deflection. Knowing the slope to be zero at the load simplifies the procedure for this problem.

The deflection Δ is then determined by taking the moment of the area of the M/EI diagram between points A and B with respect to point B.

The area of the M/EI diagram between points A and B is

$$\frac{1}{2}\left(\frac{PL}{4EI}\right)\frac{L}{2} = \frac{PL^2}{16EI}$$

The moment arm to the centroid of the triangular area from point B is $(2/3)(L/2) = L/3$. Then

$$\Delta = \left(\frac{PL^2}{16EI}\right)\left(\frac{L}{3}\right) = \frac{PL^3}{48EI}$$

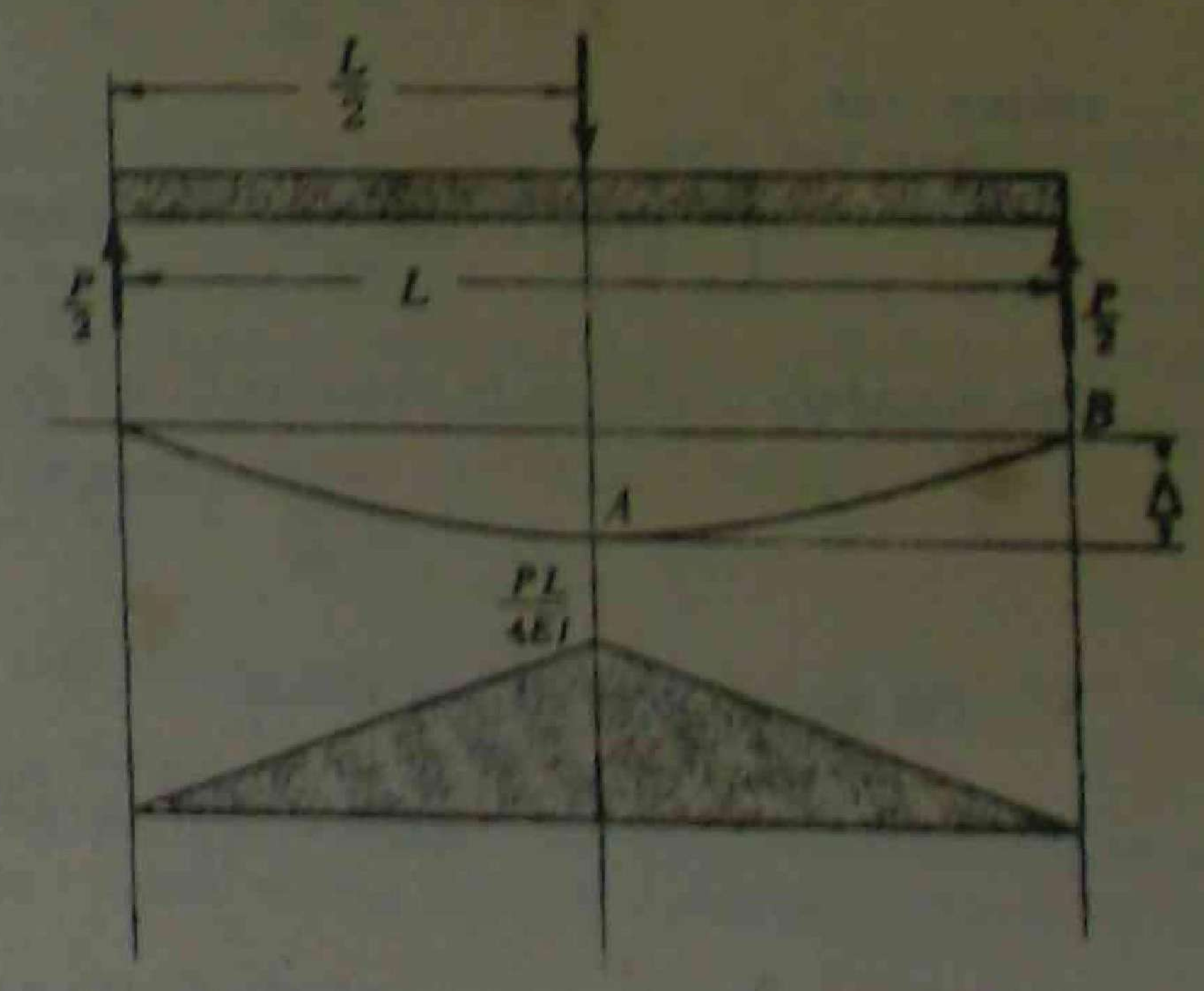


Fig. 5-11

(b) Using the conjugate beam method, sketch the conjugate beam and load it so that the intensity of load at any section is equal to the ordinate of the M/EI diagram as shown in Fig. 5-12. Consider the loads on the conjugate beam due to area A and area B as concentrated at the centroids of these areas. The magnitudes of these loads are

$$A = B = \frac{1}{2}\left(\frac{PL}{4EI}\right)\frac{L}{2} = \frac{PL^2}{16EI}$$

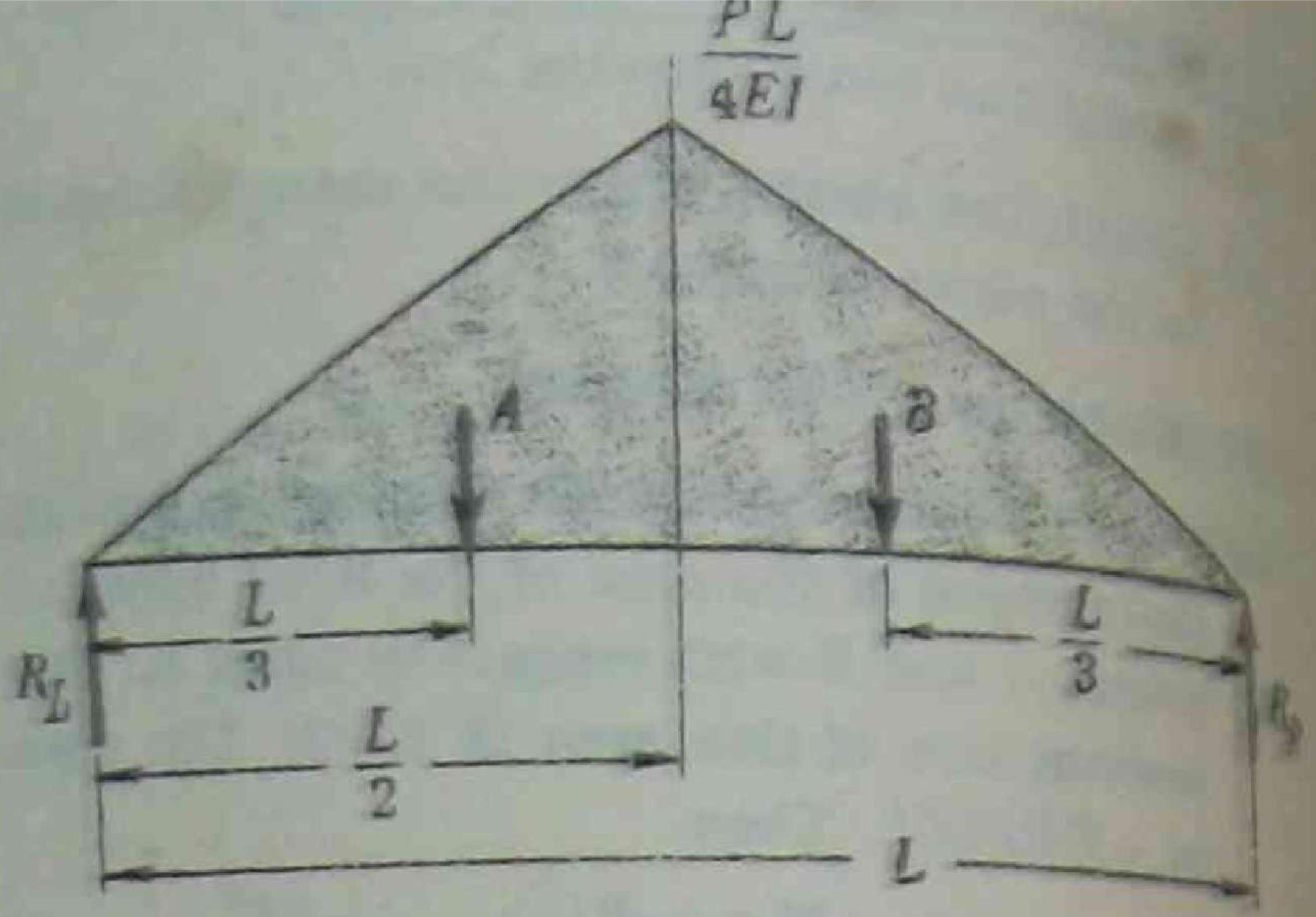


Fig. 5-12

The right and left end reactions may now be determined. In this case they will be equal due to the symmetry of loading, and may be found from a vertical summation of "loads" being equal to zero.

$$R_L = R_R = \frac{PL^2}{16EI}$$

Now the deflection at any section of the original beam is equal to the bending moment of the conjugate beam at that section. The bending moment of the conjugate beam, or the deflection of the original beam, at its midpoint is

$$M_{L/2} = \left(\frac{PL^2}{16EI}\right)\left(\frac{L}{2}\right) - \left(\frac{PL^2}{16EI}\right)\left(\frac{L}{2}-\frac{L}{3}\right) = \frac{PL^3}{48EI}$$

(c) The use of step functions in this case is simplified since the beam is taken as one of uniform cross section. We may write the moment equation as

$$EI\frac{d^2y}{dx^2} = M = \frac{P}{2}x - P\left(x-\frac{L}{2}\right)H_{L/2}$$

where the step function: $H_{L/2} = 0$ for $x < L/2$, $H_{L/2} = 1$ for $x \geqq L/2$.

The first integration yields

$$EI\frac{dy}{dx} = \frac{Px^2}{4} - \frac{P(x-L/2)^2}{2}H_{L/2} + C_1$$

noting that

$$\int_0^x P(x-L/2)H_{L/2}\,dx = PH_{L/2}\int_{L/2}^x (x-L/2)\,dx = \frac{P(x-L/2)^2}{2}H_{L/2}$$

One boundary condition to evaluate C_1 is: $dy/dx = 0$ when $x = L/2$; then $C_1 = -PL^2/16$ and

$$EI\frac{dy}{dx} = \frac{Px^2}{4} - \frac{P(x-L/2)^2}{2}H_{L/2} - \frac{PL^2}{16}$$

The second integration yields

$$EIy = \frac{Px^3}{12} - \frac{P(x-L/2)^3}{6}H_{L/2} - \frac{PL^2}{16}x + C_2$$

One boundary condition to evaluate C_2 is: $y = 0$ when $x = 0$; hence $C_2 = 0$. Then substituting $L/2$ for x, the deflection at $x = L/2$ is

$$EIy = \frac{PL^3}{96} - \frac{PL^3}{32} = \frac{-PL^3}{48} \quad \text{and} \quad y = -\frac{PL^3}{48EI}$$

(*d*) Using the theorem of Castigliano, we may write the strain energy equation

$$U = \int \frac{M^2\,dx}{2EI} = \int_0^{L/2} \frac{M^2\,dx}{2EI} + \int_{L/2}^{L} \frac{M^2\,dx}{2EI} = \int_0^{L/2} \frac{(Px/2)^2\,dx}{2EI} + \int_{L/2}^{L} \frac{(Px/2 - Px + PL/2)^2\,dx}{2EI}$$

from which $U = \dfrac{P^2L^3}{96EI}$. The deflection at the load P is $\dfrac{\partial U}{\partial P} = \dfrac{PL^3}{48EI}$.

5. Determine the vertical deflection, due to bending, of point P of a cantilever beam loaded with a horizontal load F, as shown in Fig. 5-13(*a*). Neglect deformation of the vertical member.

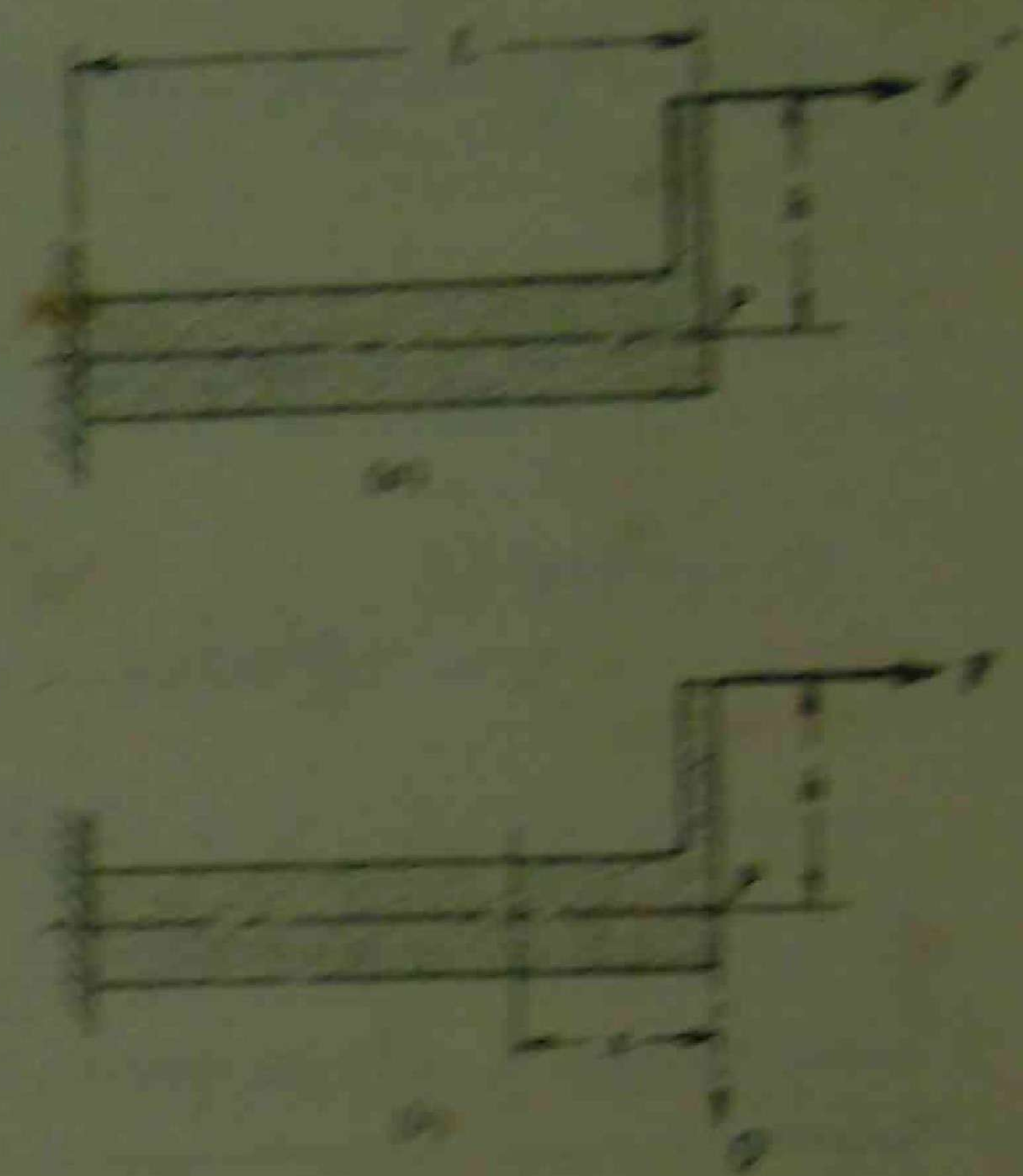


Fig. 5-13

Solution:

A problem of this type may be solved by the theorem of Castigliano if we superimpose a vertical load Q at the point where the deflection is to be found, as shown in Fig. 5-13(*b*). The strain energy is

$$U = \int_0^L \frac{(Fa + Qx)^2\,dx}{2EI}$$

$$= \frac{F^2a^2L + FaQL^2 + Q^2L^3/3}{2EI}$$

The vertical deflection is

$$\delta_v = \frac{\partial U}{\partial Q} = \frac{FaL^2 + 2QL^3/3}{2EI}$$

Now setting $Q = 0$, we have the vertical deflection at the point P:

$$\delta_v = \frac{FaL^2}{2EI}$$

6. A shaft is supported by two anti-friction bearings at A and C with loads of 30 lb each acting at points B and F, as shown in Fig. 5-14 below. The portion of the shaft between B and C has a diameter of $2D$ compared to a diameter of D for the portions of the shaft between A and B and between C and F. In connection with a critical speed calculation, it is required to determine the deflection of the shaft at points B and F. Use the area moment method and consider deflections due to bending only. (See the chapter on Shafts for a more extensive deflection analysis.)

Solution:

(*1*) Sketch the elastic curve passing through points of zero deflection at points A and C, and draw the tangent at point A.

(2) Sketch the M/EI diagram noting that the moment of inertia of the section having a diameter $2D$ is 16 times that of the sections having a diameter D. The I refers to the smaller diameter portions of the shaft.

(*3*) Determine Δ_1 by taking the moment of the areas of the M/EI diagram between points A and F with respect to point F.

$$\Delta_1 = \left(\frac{180}{EI}\right)\left(\frac{6}{2}\right)(4) + \left(\frac{180}{16EI}\right)(6)(9) + \left(\frac{180}{EI}\right)\left(\frac{6}{2}\right)(14) = \frac{10{,}327.5}{EI}$$

(4) Determine Δ_2 by taking moment of the areas of the M/EI diagram between points A and C with respect to point C.

$$\Delta_2 = \left(\frac{180}{16EI}\right)(6)(3) + \left(\frac{180}{EI}\right)\left(\frac{6}{2}\right)(8) = \frac{4522.5}{EI}.$$

(5) Determine Δ_3 by proportion:

$$\Delta_3 = \frac{18}{12}\Delta_2 = \frac{6783.75}{EI}.$$

Then $y_1 = \Delta_1 - \Delta_3 = \dfrac{3543.75}{EI}$.

(6) Determine Δ_5 by proportion:

$$\Delta_5 = \frac{\Delta_3}{3} = \frac{2261.25}{EI}.$$

(7) Determine Δ_4:

$$\Delta_4 = \left(\frac{180}{EI}\right)\left(\frac{6}{2}\right)(2) = \frac{1080}{EI}.$$

Then $y_2 = \Delta_5 - \Delta_4 = \dfrac{1181.25}{EI}$.

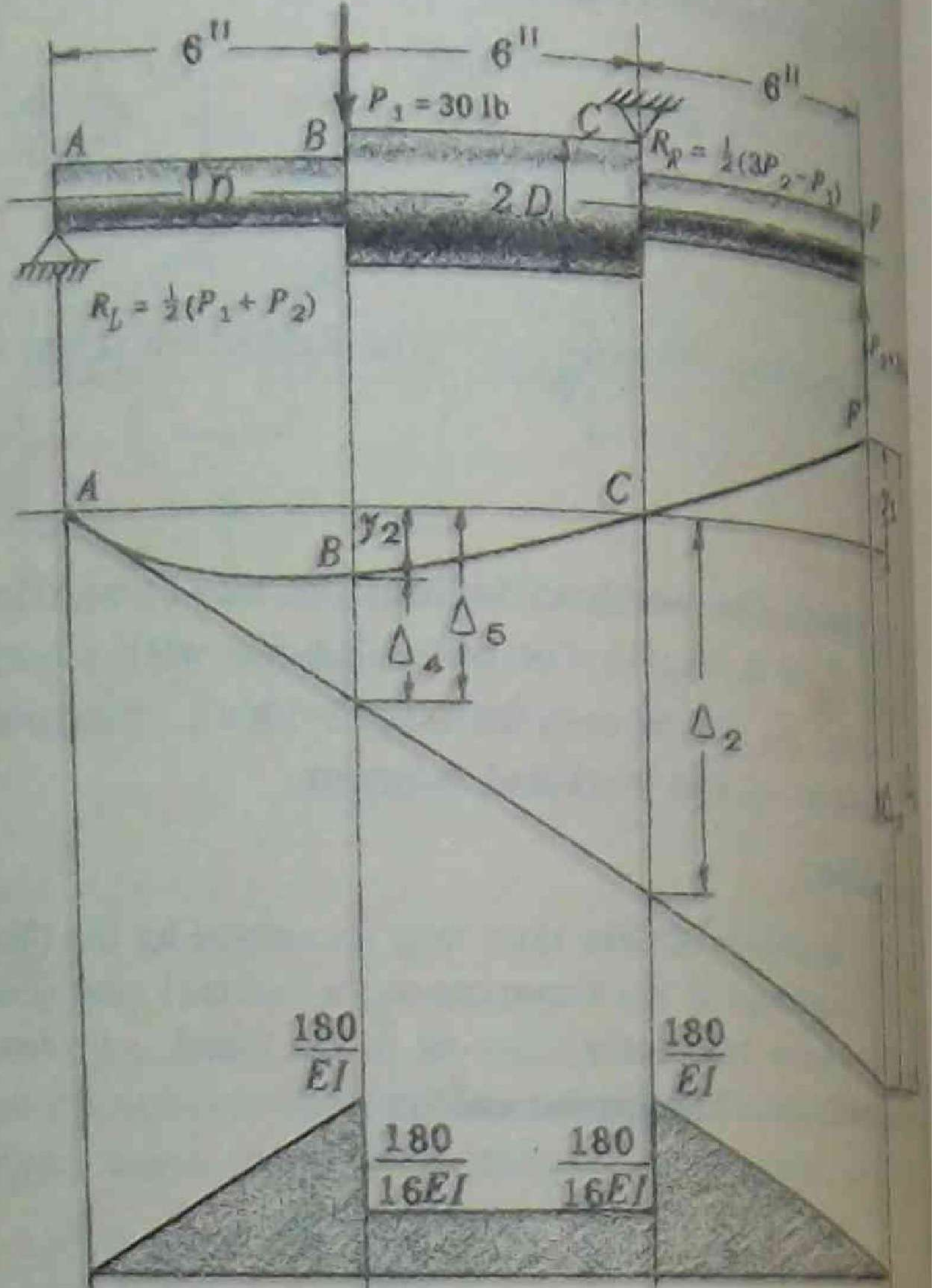


Fig. 5-14

7. Solve Problem 6 using the theorem of Castigliano.

Solution:

The total strain energy U due to bending is

$$U = \int \frac{M^2\,dx}{2EI} = U_1 + U_2 + U_3$$

where U_1 = energy from $x = 0$ to $x = 6$
U_2 = energy from $x = 6$ to $x = 12$
U_3 = energy from $x = 12$ to $x = 18$.

$$U_1 = \int_0^6 \frac{(R_L x)^2\,dx}{2EI} = \left[\frac{R_L^2 x^3}{6EI}\right]_0^6 = \frac{36R_L^2}{EI} = \frac{36}{EI}\left(\frac{P_1 + P_2}{2}\right)^2$$

Now evaluate U_3:
$$U_3 = \int_{12}^{18} \frac{P_2^2(18-x)^2\,dx}{2EI} = \frac{36P_2^2}{EI}$$

If I is the moment of inertia of a section of diameter D, then the moment of inertia of the section $x = 6$ to $x = 12$, which has a diameter of $2D$, is $16I$. The strain energy in the center portion is

$$U_2 = \int_6^{12} \frac{[R_L x - P_1(x-6)]^2\,dx}{2E(16I)}$$

Integrating, substituting $\frac{1}{2}(P_1 + P_2)$ for R_L and collecting terms,

$$U_2 = \frac{72}{32EI}\left[7\left(\frac{P_2 - P_1}{2}\right)^2 + 9P_1\left(\frac{P_1 + P_2}{2}\right) - 6P_1^2\right]$$

The total strain energy is $U = U_1 + U_2 + U_3$

$$U = \frac{36}{EI}\left(\frac{P_1+P_2}{2}\right)^2 + \frac{72}{32EI}\left[7\left(\frac{P_2-P_1}{2}\right)^2 + 9P_1\left(\frac{P_1+P_2}{2}\right) - 9P_1^2\right] + \frac{36P_1^2}{EI}$$

The deflection under P_1 is

$$\frac{\partial U}{\partial P_1} = \frac{18}{EI}(P_1+P_2) + \frac{9}{4EI}(P_2+\frac{P_1}{2}) = \frac{1080}{EI} + \frac{101.25}{EI} = \frac{1181.25}{EI}$$

The deflection at P_2 is

$$\frac{\partial U}{\partial P_2} = \frac{18}{EI}(P_1+P_2) + \frac{72}{32EI}\left[14\left(\frac{P_2-P_1}{2}\right)\left(\frac{1}{2}\right) + \frac{9P_1}{2}\right] + \frac{72P_2}{EI} = \frac{1080}{EI} + \frac{303.75}{EI} + \frac{2160}{EI} = \frac{3543.75}{EI}$$

1. (a) Using the area moment method, show that the maximum bending deflection of a simply supported member of length L and carrying a uniformly distributed load of w lb per inch, as shown in Fig. 5-15(a), is $5wL^4/384EI$. (b) Same as (a) except use Castigliano's theorem.

Solution:

(a) Using the area moment method.

1. Sketch the elastic curve as shown in Fig. 5-15(b). Draw a horizontal tangent to the elastic curve at point M. Then the deflection y is the moment of the area of the moment diagram between points M and B, with respect to B, divided by EI.

2. Sketch the moment diagram as shown in Figure 5-15(c).

3. For a change, let us draw the moment diagram by parts, Fig. 5-15(d), to illustrate such a procedure. This procedure in some cases might be expedient in order to simplify the determination of areas and centroids of areas.

4. In this case we are only concerned with the moments of sections I and II.

The area of section I $= (\frac{wL^2}{4})(\frac{L}{4}) = \frac{wL^3}{16}$

The area of section II $= (\frac{wL^2}{8})(\frac{L}{2})(\frac{1}{3}) = \frac{wL^3}{48}$

The distance of the centroid of section I to point B is $(2/3)(L/2) = L/3$.

The distance of the centroid of section II to point B is $(3/4)(L/2) = 3L/8$.

Then $EIy = (\frac{wL^3}{16})(\frac{L}{3}) - (\frac{wL^3}{48})(\frac{3L}{8})$

$$= \frac{5wL^4}{384}$$

and $$y = \frac{5wL^4}{384EI}$$

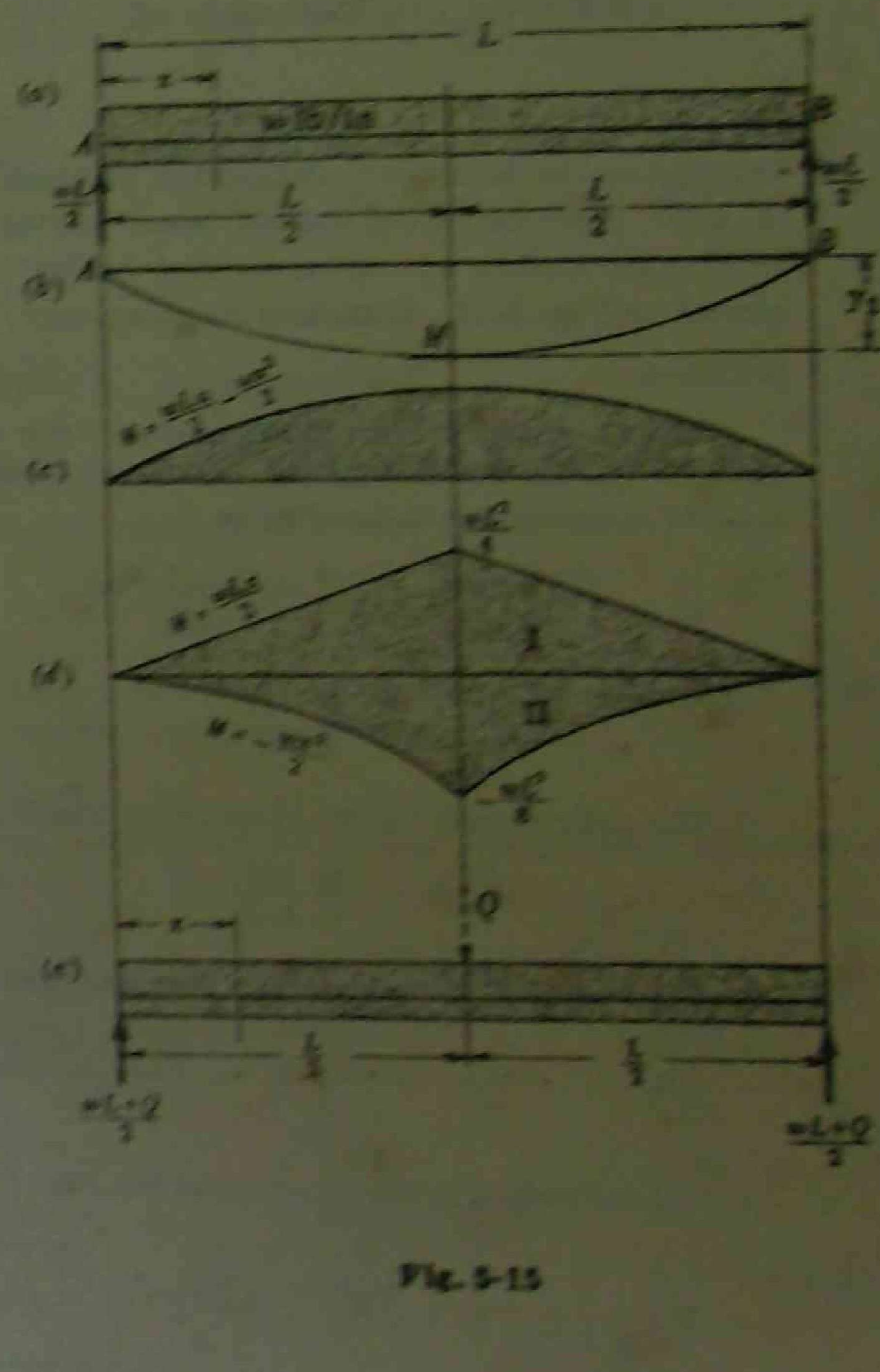


Fig. 5-15

Chapter 7

Machine Vibrations

VIBRATORY MOTIONS in machinery arise when variable forces act on elastic parts. Usually these motions are undesirable although in some cases (vibratory conveyors, for example) they are deliberately designed into the machine.

ANALYSING VIBRATIONS requires the following general procedure:

1. Evaluating masses and elasticity of parts involved.
2. Estimating amount of friction involved.
3. Idealizing the actual mechanical device, replacing it by an approximately equivalent system of masses, springs, and dampers.
4. Writing differential equations of motion for the idealized system.
5. Solving the equations and interpreting the results.

THE SIMPLEST IDEAL SYSTEM consists of a single mass, single spring, and a dashpot, as shown in Fig. 7-1. The differential equation of motion for this system is

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

where

- m = the mass.
- k = the spring constant (force per unit deflection).
- c = the damping (frictional) constant (force per unit of velocity). (Viscous damping, in which the resisting force is proportional to velocity, is assumed.)
- $F(t)$ = any external force, a function of time.
- x = the displacement of the mass from the static equilibrium position.
- $\dot{x}$, $\ddot{x}$ = derivatives, first and second respectively, of x with respect to t.

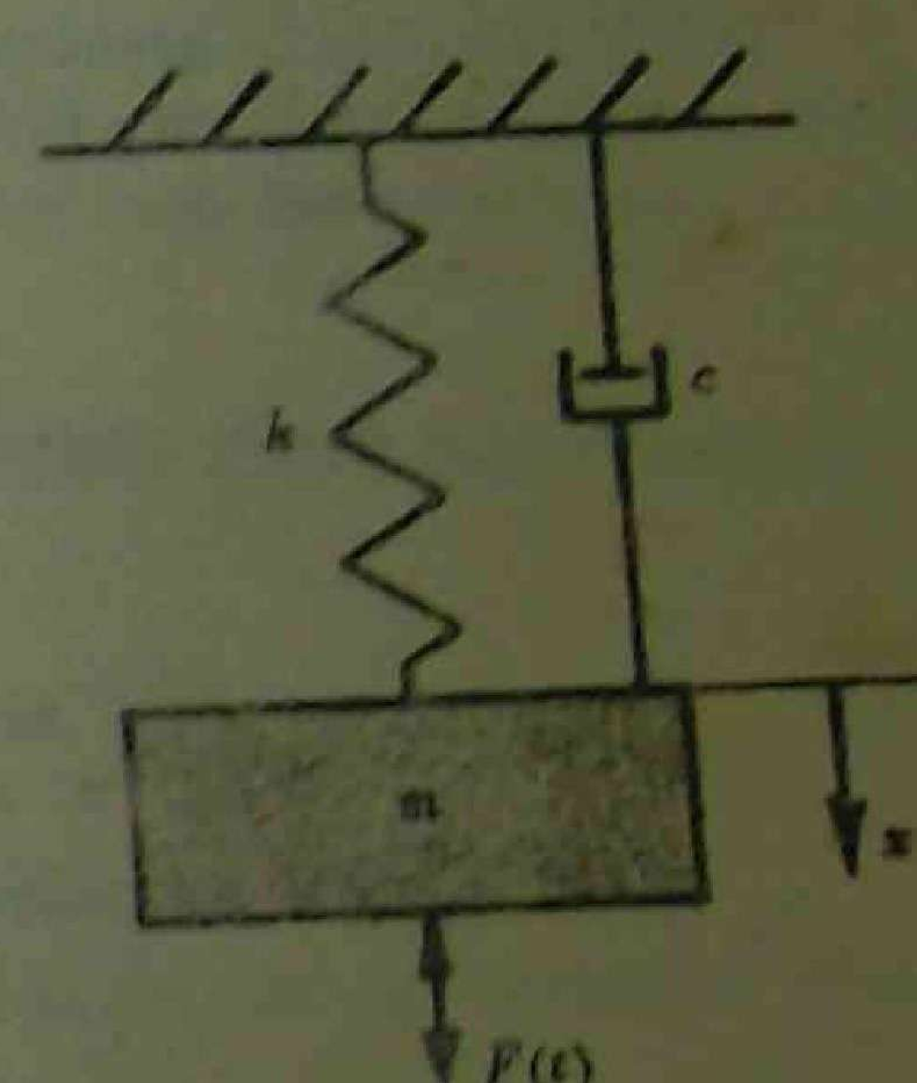


Fig. 7-1

ANY SINGLE-DEGREE-OF-FREEDOM SYSTEM can be described by the same form of differential equation as written above, if the restoring force (spring force) is proportional to the displacement and if the frictional force is proportional to the velocity. For the general single-degree-of-freedom system we shall write

$$m_e\ddot{x} + c_e\dot{x} + k_e x = F(t)$$

where m_e, c_e, k_e are respectively the equivalent mass, damping constant, and spring constant. The displacement x may be either linear or angular.

The forcing function, $F(t)$, may in practice be of any form. For this analysis it is assumed to be sinusoidal:

$$F(t) = F_o \sin \omega t$$

where F_o is the amplitude of the externally applied force and ω is the frequency.

FREE VIBRATIONS may occur when, after an initial disturbance, no external forcing function is present, i.e. $F(t) = 0$. The differential equation is simply

$$m_e \ddot{x} + c_e \dot{x} + k_e x = 0$$

The solution of this equation can be written

$$x = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where $s_1 = -\frac{c_e}{2m_e} + \sqrt{(\frac{c_e}{2m_e})^2 - \frac{k_e}{m_e}}$ and $s_2 = -\frac{c_e}{2m_e} - \sqrt{(\frac{c_e}{2m_e})^2 - \frac{k_e}{m_e}}$

and A_1 and A_2 are constants determined by the initial conditions.

In the special case where $(c_e/2m_e)^2 = k_e/m_e$, $s_1 = s_2 = s$ and the solution is $x = (A + Bt)e^{-st}$

CRITICAL DAMPING refers to the special case just mentioned for which $(\frac{c_e}{2m_e})^2 = \frac{k_e}{m_e}$, and $c_e = (c_e)_c = 2\sqrt{k_e m_e}$ is called the **critical value** of the damping coefficient.

If the damping is greater than critical, then the solution of the differential equation for free vibration contains no periodic terms. The mass, after an initial disturbance, returns toward the equilibrium position but does not oscillate.

DAMPING LESS THAN CRITICAL. This is the oscillatory situation. The solution of the differential equation for free vibration can be written in the form

$$x = e^{-\alpha t} X \sin(\omega_d t + \gamma) \quad \text{where} \quad \alpha = \frac{c_e}{2m_e}, \quad \omega_d = \sqrt{\frac{k_e}{m_e} - (\frac{c_e}{2m_e})^2}.$$

ω_d is the **damped frequency** of the system. If damping were zero the frequency would be $\omega_n = \sqrt{\frac{k_e}{m_e}}$ which is called the **natural frequency**.

The constants X and γ are determined by the initial conditions.

FOR FORCED VIBRATIONS, the solution of the differential equation is that given above for free vibrations plus a particular integral. The solution can be written in the form

$$x = e^{-\alpha t} X \sin(\omega_d t + \gamma) + Y \sin(\omega t - \phi)$$

The first part of the above expression represents the **transient** vibration; this dies out with time. The second part is called the **steady state** vibration and is the part which is usually of most interest to the engineer.

THE STEADY STATE AMPLITUDE Y is $Y = \frac{F_o}{\sqrt{(k_e - m_e\omega^2)^2 + (c_e\omega)^2}}$. This can be written

$$Y = \frac{(F_o/k)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

where $r = \omega/\omega_n$ is the frequency ratio, and $\xi = c_e/(c_e)_c$ is the damping ratio.

11. A steel latch is $\frac{1}{4}$" thick. A force P of 600 pounds is uniformly distributed as shown in Fig. 2-23. Determine the maximum shear, tensile, and compressive stresses at section A-A and at point B.

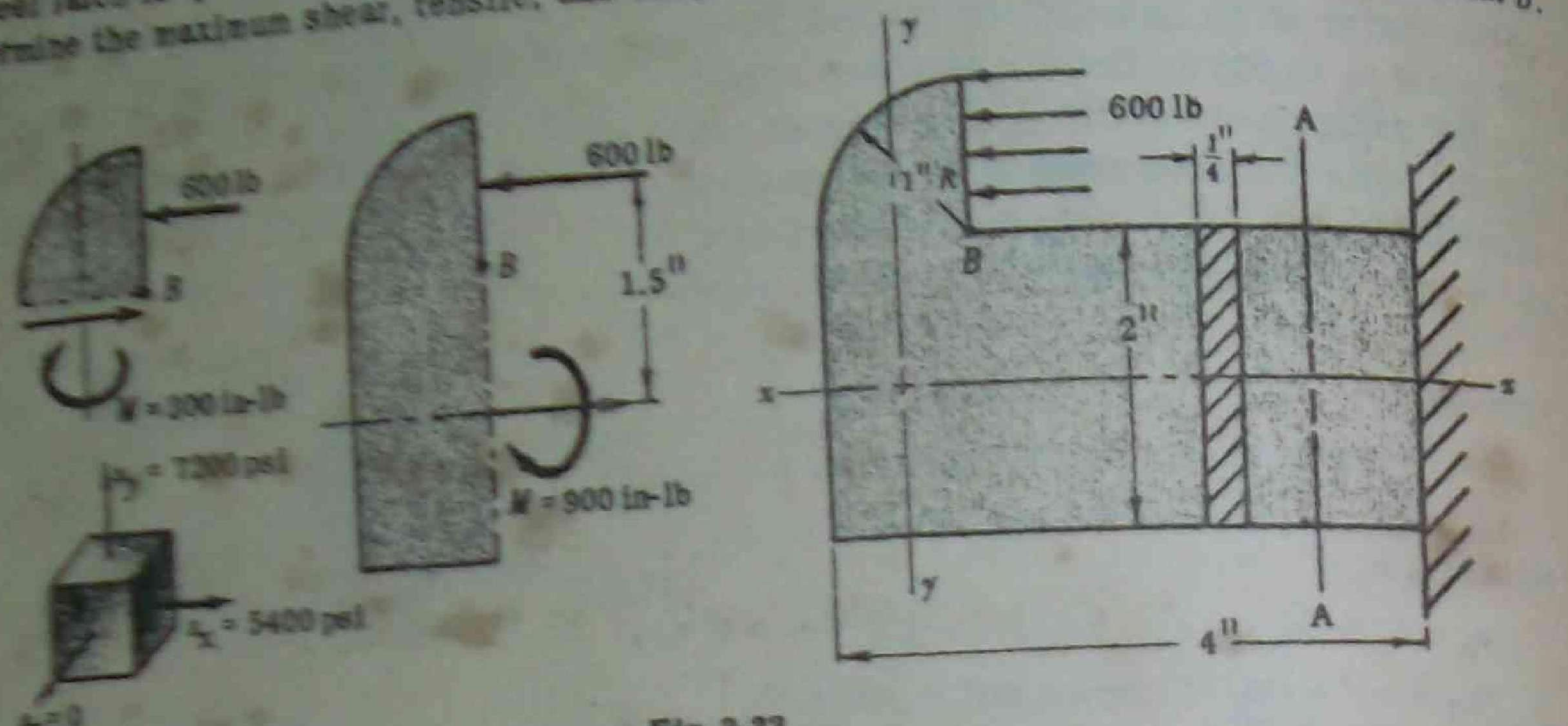


Fig. 2-23

Solution:

At section A-A:

The critical point is at the top fibers.

$$s_x = \frac{Mc}{I} + \frac{P}{A} = \frac{(600 \times 1.5)(1)(12)}{(0.25)(2)^3} + \frac{600}{(0.25)(2)} = 6600 \text{ psi}$$

$s_n(\text{max}) = s_x = 6600$ psi (tension) at top fibers of section A-A.

$s_n(\text{min}) = -4200$ psi (compression) at bottom fibers of section A-A.

$\tau(\text{max}) = 6600/2 = 3300$ psi (shear) at top fibers of section A-A.

At point B (neglecting stress concentration):

$$s_x = \frac{Mc}{I} + \frac{P}{A} = \frac{(900)(1)(12)}{(0.25)(2)^3} + \frac{600}{(0.25)(2)} = 6600 \text{ psi (tension)}$$

$$s_y = \frac{Mc}{I} = \frac{(300)(0.5)(12)}{(0.25)(1)^3} = 7200 \text{ psi (tension)}$$

$s_z = 0, \quad \tau_{xy} = 0$

$$s_n(\text{max}) = \frac{6600 + 7200}{2} + \sqrt{\left(\frac{6600 - 7200}{2}\right)^2 - 0} = 7200 \text{ psi (tension)}$$

$$s_n(\text{min}) = \frac{6600 + 7200}{2} - \sqrt{\left(\frac{6600 - 7200}{2}\right)^2 - 0} = 6600 \text{ psi (tension)}$$

$$\tau(\text{max}) = \frac{s_n(\text{max}) - 0}{2} = 3600 \text{ psi (shear)}$$

12. Determine the maximum normal stress and the maximum shear stress at section A-A for the crank shown in Fig. 2-24 when a load of 2000 lb is assumed to be concentrated at the center of the crank pin.

Solution:

The critical points are at the front and back fibers of the section.

$M = (2000)(3.5) = 7000$ in-lb

$T = (2000)(5) = 10,000$ in-lb

$$s_x = \frac{Mc}{I} = \frac{(7000)(1.5)(64)}{\pi(3^4)} = 2640 \text{ psi}$$

$$s_{xy} = \frac{Tr}{J} = \frac{(10,000)(1.5)(32)}{\pi(3^4)} = 1885 \text{ psi}$$

$s_n(\text{max}) = 2640/2 + \sqrt{(2640/2)^2 + (1885)^2} = 3620$ psi (tension)

$\tau(\text{max}) = \sqrt{(2640/2)^2 + (1885)^2} = 2300$ psi (shear)

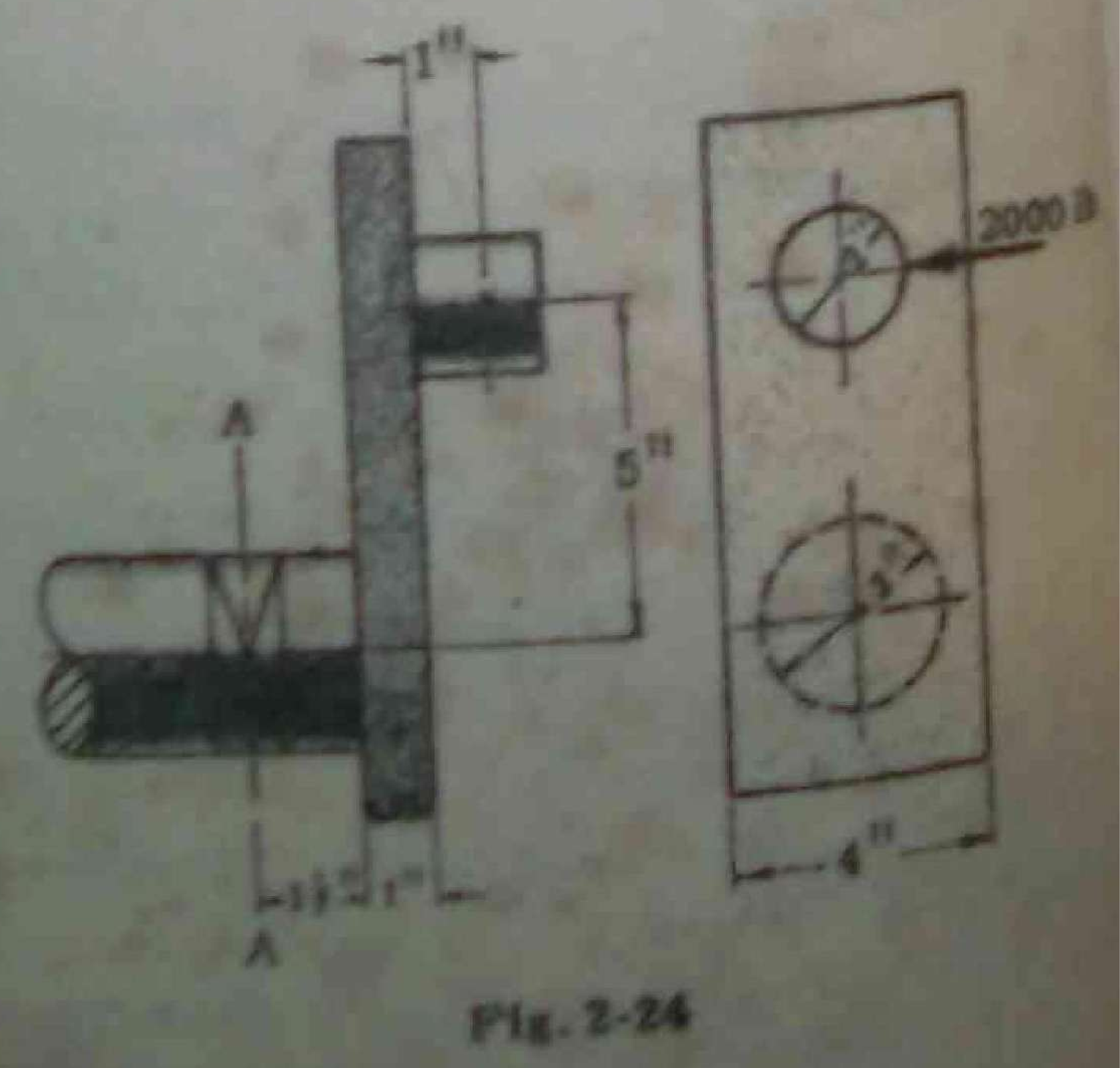


Fig. 2-24

MAGNIFICATION FACTOR M is

$$M = \frac{Y}{F_o/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

M is the ratio of steady state displacement amplitude to the displacement which would be caused by a static force equal to F_o.

THE PHASE ANGLE ϕ can be determined from the following.

$$\tan\phi = \frac{c_e\omega}{k_e - m_e\omega^2}, \qquad \sin\phi = \frac{c_e\omega}{\sqrt{(k_e - m_e\omega^2)^2 + (c_e\omega)^2}}$$

THE FORCE TRANSMITTED TO THE BASE is the sum of the spring force and the damping force:

$$k_e x + c_e \dot{x}$$

Using the previously displayed steady state solution for x it can be shown that the transmitted force has the amplitude

$$F_{TR} = \frac{F_o\sqrt{k_e^2 + (c_e\omega)^2}}{\sqrt{(k_e - m_e\omega^2)^2 + (c_e\omega)^2}}$$

TRANSMISSIBILITY RATIO is the ratio of the amplitude of the transmitted force to the amplitude it would have if the mass were bolted to the base (no spring or damper).

$$\text{T.R.} = \frac{F_{TR}}{F_o} = \frac{\sqrt{k_e^2 + (c_e\omega)^2}}{\sqrt{(k_e - m_e\omega^2)^2 + (c_e\omega)^2}}$$

$$= \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

THE FORCING FUNCTION, in the previous discussion, was in the form of a periodic force applied to the moving mass. Another important situation is illustrated in Fig. 7-2. Here a periodic motion of the base induces motion of the mass. The usual design problem in this situation is to choose spring and damper such that the amplitude of motion of the mass will be small compared to the amplitude of motion of the base.

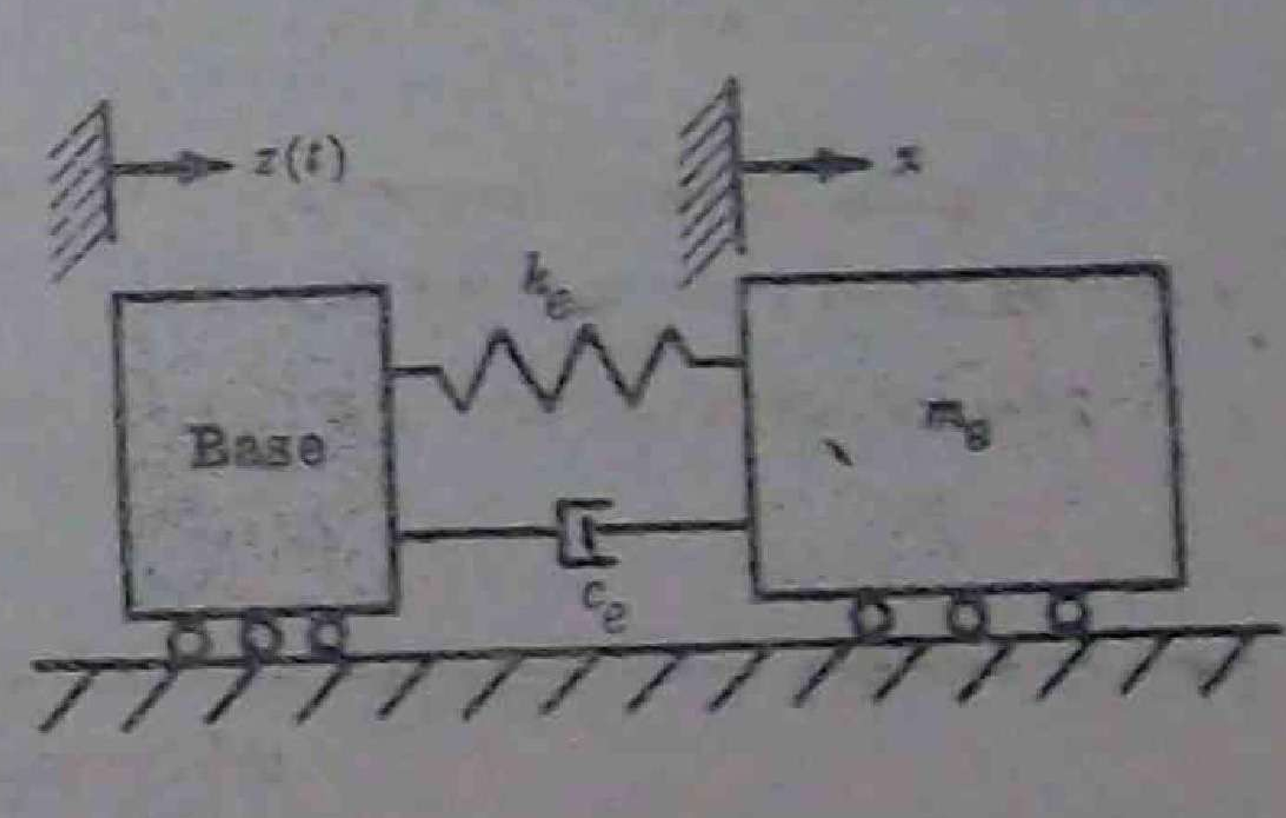


Fig. 7-2

If $z(t)$ is taken to be sinusoidal, i.e.

$$z(t) = z\sin\omega t$$

then the differential equation for motion of the mass is

$$m_e\ddot{x} + c_e\dot{x} + k_e x = z\sqrt{k_e^2 + (c_e\omega)^2}\,\sin(\omega t - \psi)$$

where ψ is a phase angle.

$$\cos\psi = \frac{k_e}{\sqrt{k_e^2 + (c_e\omega)^2}}, \qquad \sin\psi = \frac{-c_e\omega}{\sqrt{k_e^2 + (c_e\omega)^2}}$$

The above differential equation, except for the phase angle ψ, is identical in form with the equation previously discussed. Solution will show that the amplitude of the steady state vibration of the mass is

$$Y = \frac{z\sqrt{k_g^2 + (c_g\,\omega)^2}}{\sqrt{(k_g - m_g\,\omega^2)^2 + (c_g\,\omega)^2}}$$

TRANSMISSIBILITY RATIO is the ratio of amplitude of the motion of the mass to that of the base.

$$\text{T.R.} = \frac{Y}{z} = \frac{\sqrt{k_g^2 + (c_g\,\omega)^2}}{\sqrt{(k_g - m_g\,\omega^2)^2 + (c_g\,\omega)^2}}$$

This is identical with the force transmissibility ratio previously discussed.

SYSTEMS WITH MORE THAN ONE DEGREE OF FREEDOM cannot be described by a single second order differential equation. A complete analysis of such a system would, in general, require the simultaneous solution of a set of n second order equations, where n is the number of degrees of freedom of the system. However, relatively simple practical means are available for determining the lowest (or fundamental) frequency of vibration. This one piece of information is of great value to the design engineer.

The two-degree-of-freedom system of Fig. 7-3 has two modes of vibration. In the first mode the two masses will move in phase, reaching maximum displacement in the same direction at the same time. In the second mode the two masses will be out of phase, reaching maximum displacements in opposite directions at the same time.

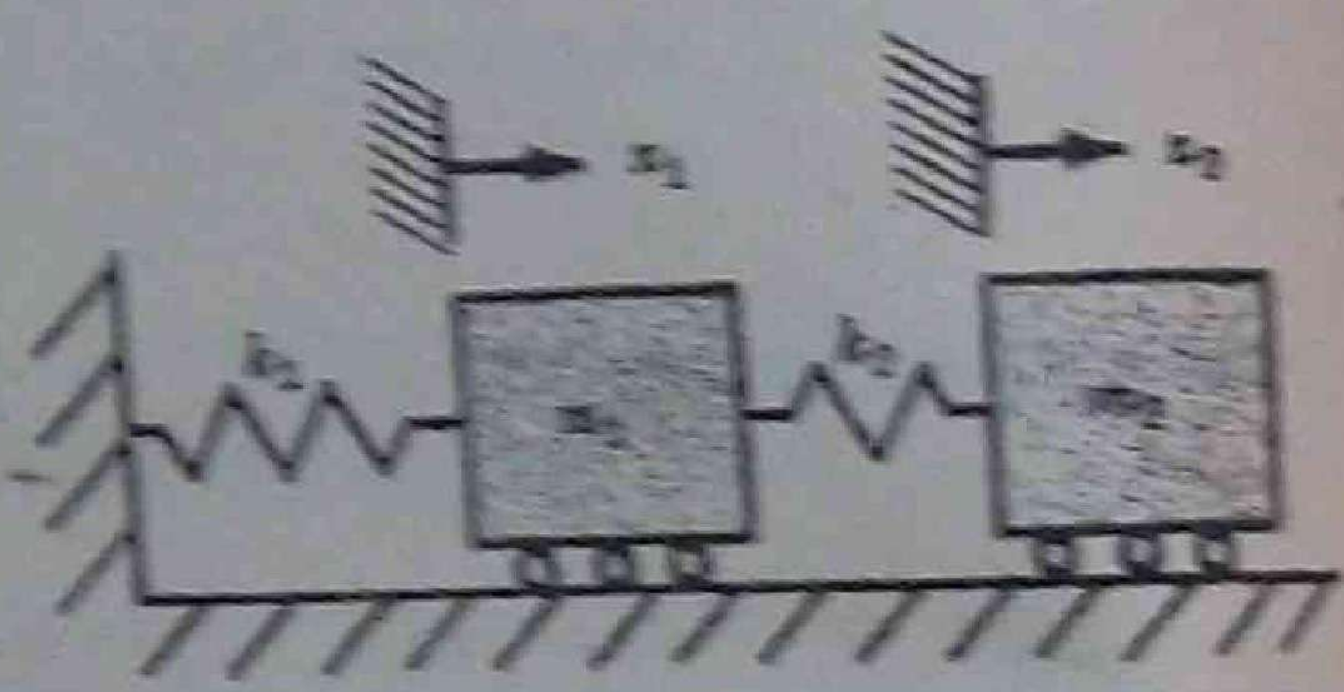


Fig. 7-3

THE ENERGY METHOD for determining the first mode frequency is based on the idea that, neglecting friction, the maximum kinetic energy of the system must equal the maximum potential energy.

Let X_1 = amplitude of displacement of mass m_1, and X_2 = amplitude of displacement of mass m_2. Assume sinusoidal motion of frequency ω.

The maximum kinetic energy of the system will be

$$\text{Max. K.E.} = \tfrac{1}{2} m_1 X_1^2 \omega^2 + \tfrac{1}{2} m_2 X_2^2 \omega^2$$

The maximum potential energy stored in the springs will be

$$\text{Max. P.E.} = \tfrac{1}{2} k_1 X_1^2 + \tfrac{1}{2} k_2 (X_2 - X_1)^2$$

Neglecting friction, $\quad$ Max. K.E. = Max. P.E.

from which $\quad \omega^2 = \dfrac{k_1 X_1^2 + k_2 (X_2 - X_1)^2}{m_1 X_1^2 + m_2 X_2^2} \quad$ or $\quad \omega^2 = \dfrac{k_1 + k_2 (X_2/X_1 - 1)^2}{m_1 + m_2 (X_2/X_1)^2}$

This equation would give us directly the first, or lowest, natural frequency of vibration if we knew the ratio of amplitudes X_2/X_1. The practical procedure is to try a series of values for this ratio. The value which gives the lowest result for ω is the most nearly correct.

RESONANCE is variously defined in different textbooks. The term refers generally to operation in the vicinity of maximum forced vibration amplitude. For a frictionless system the means operation at the natural frequency $\omega_n = \sqrt{k_e/m_e}$.

With viscous damping and a forcing function of the form $F_o \sin \omega t$ applied to the mass, maximum amplitude is obtained when the operating frequency ω is

$$\omega_{max\,\Gamma} = \omega_n\sqrt{1 - 2\xi^2}$$

Notice that this is different from the damped frequency ω_d.

$$\omega_d = \omega_n\sqrt{1 - \xi^2}$$

In the absence of deliberately built-in damping devices, the factor $\xi = c_e/(c_e)_c$ is usually small enough that ω_n, ω_d and $\omega_{max\,\Gamma}$ are very nearly equal. Hence ω_n is ordinarily used for engineering estimates. In the problems to follow, when *resonance* is mentioned, it will mean operation at the natural frequency ω_n.

For a multi-degree-of-freedom system, *resonance* will mean operation at any one of the natural frequencies.

SOLVED PROBLEMS

1. Write the differential equation for the free vibration of the system shown in Fig. 7-4, x being measured from the unstressed spring position.

Solution:

We first make a freebody sketch of the mass and carefully label all forces acting in the x direction. We then apply Newton's second law, setting the sum of the external forces equal to the product of mass and acceleration.

$$-c\dot{x} - kx = m\ddot{x} \qquad \text{or} \qquad m\ddot{x} + c\dot{x} + kx = 0$$

Note that the spring force is properly written $-kx$, because it is opposite in sense to x. Likewise, the damping force is written $-c\dot{x}$ because it is opposite in sense to the velocity $\dot{x}$.

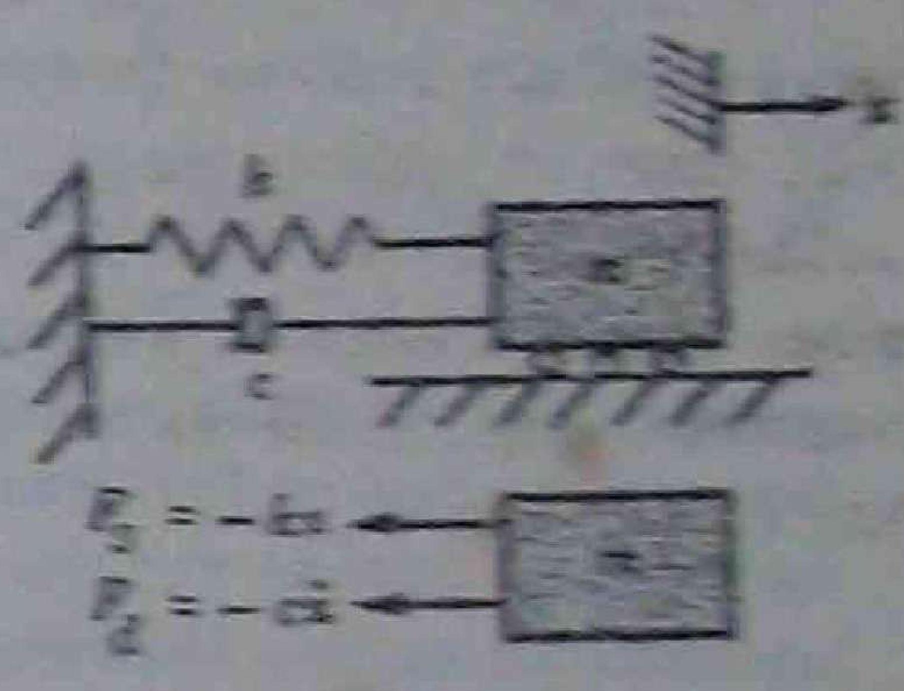


Fig. 7-4

2. Write the differential equation for the free vibration of the system shown in Fig. 7-5. Neglect the mass of the lever.

Solution:

An angular motion is involved. We shall sum moments of external forces around pivot O and set this equal to the product of angular acceleration and moment of inertia with respect to the pivot.

For a small displacement θ, the spring force is very nearly $-ka\theta$ and the damping force $-ca\dot{\theta}$. Also, the moment arms for these forces are very nearly equal to a. The moment arm for the weight force is $b \sin\theta$, which will be approximated as $b\theta$. The moment of inertia of the mass with respect to the pivot O is mb^2; hence

$$-(ca\dot{\theta})a - (ka\theta)a - mg(b\theta) = mb^2\ddot{\theta}$$

or

$$mb^2\ddot{\theta} + ca^2\dot{\theta} + (ka^2 + mbg)\theta = 0$$

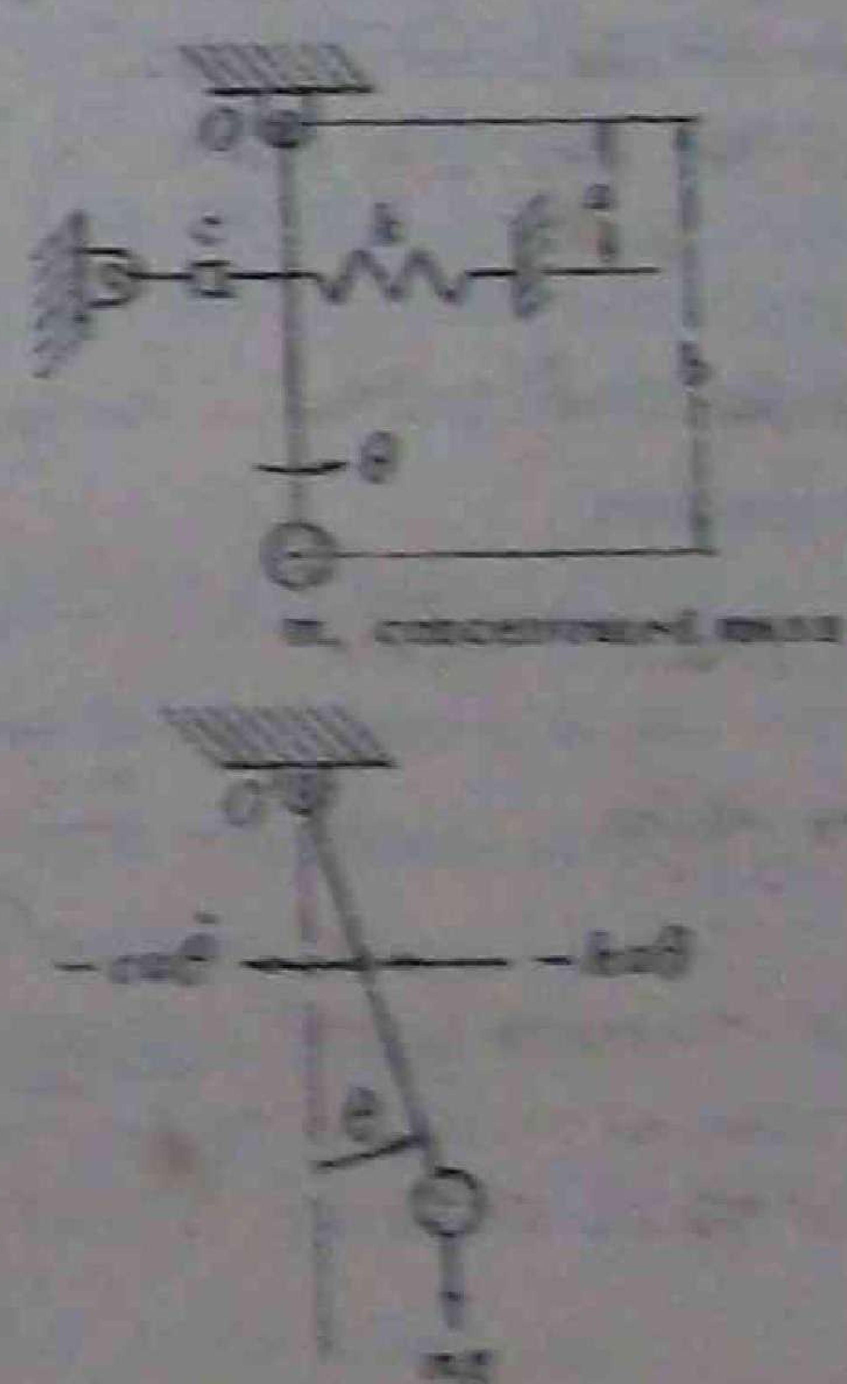


Fig. 7-5

3. For the system of Problem 2 determine (a) the natural frequency, (b) the damped frequency, (c) c_c, the critical value of damping factor c.

Solution:

Comparing the equation written in Problem 2 with the general single-degree-of-freedom equation discussed earlier, we have

$$x = \theta, \quad \dot{x} = \dot{\theta}, \quad \ddot{x} = \ddot{\theta}, \quad m_e = mb^2, \quad c_e = ca^2, \quad k_e = ka^2 + mgb$$

Hence

(a) $\omega_n = \sqrt{\dfrac{k_e}{m_e}} = \sqrt{\dfrac{ka^2 + mgb}{mb^2}}$ (b) $\omega_d = \sqrt{\dfrac{k_e}{m_e} - (\dfrac{c_e}{2m_e})^2} = \sqrt{\dfrac{ka^2 + mgb}{mb^2} - (\dfrac{ca^2}{2mb^2})^2}$

(c) $(c_e)_c = c_c a^2 = 2\sqrt{k_e m_e} = 2\sqrt{(ka^2 + mgb)mb^2}$ or $c_c = (2/a^2)\sqrt{(ka^2 + mgb)mb^2}$

4. Write the differential equation for the system of Fig. 7-6.

Solution:

Again we assume small displacements and make the same approximations as in Problem 2. We define θ to be measured from the **static equilibrium position**. This means that the spring force must initially be large enough to balance the effect of the weight. Taking the moments around pivot O,

$$(-c a\dot{\theta})a + (-ka\theta - \frac{b}{a}mg)a + mgb + F_0 b \sin \omega t = mb^2\ddot{\theta}$$

or

$$mb^2\ddot{\theta} + ca^2\dot{\theta} + ka^2\theta = F_0 b \sin \omega t$$

Notice that, with θ measured from the static equilibrium position, the weight force drops out. Although this system is the same as that of Problems 2 and 3, except for the orientation with respect to vertical, the behavior is different. For example, the natural frequency for this system is

$$\omega_n = (a/b)\sqrt{k/m}$$

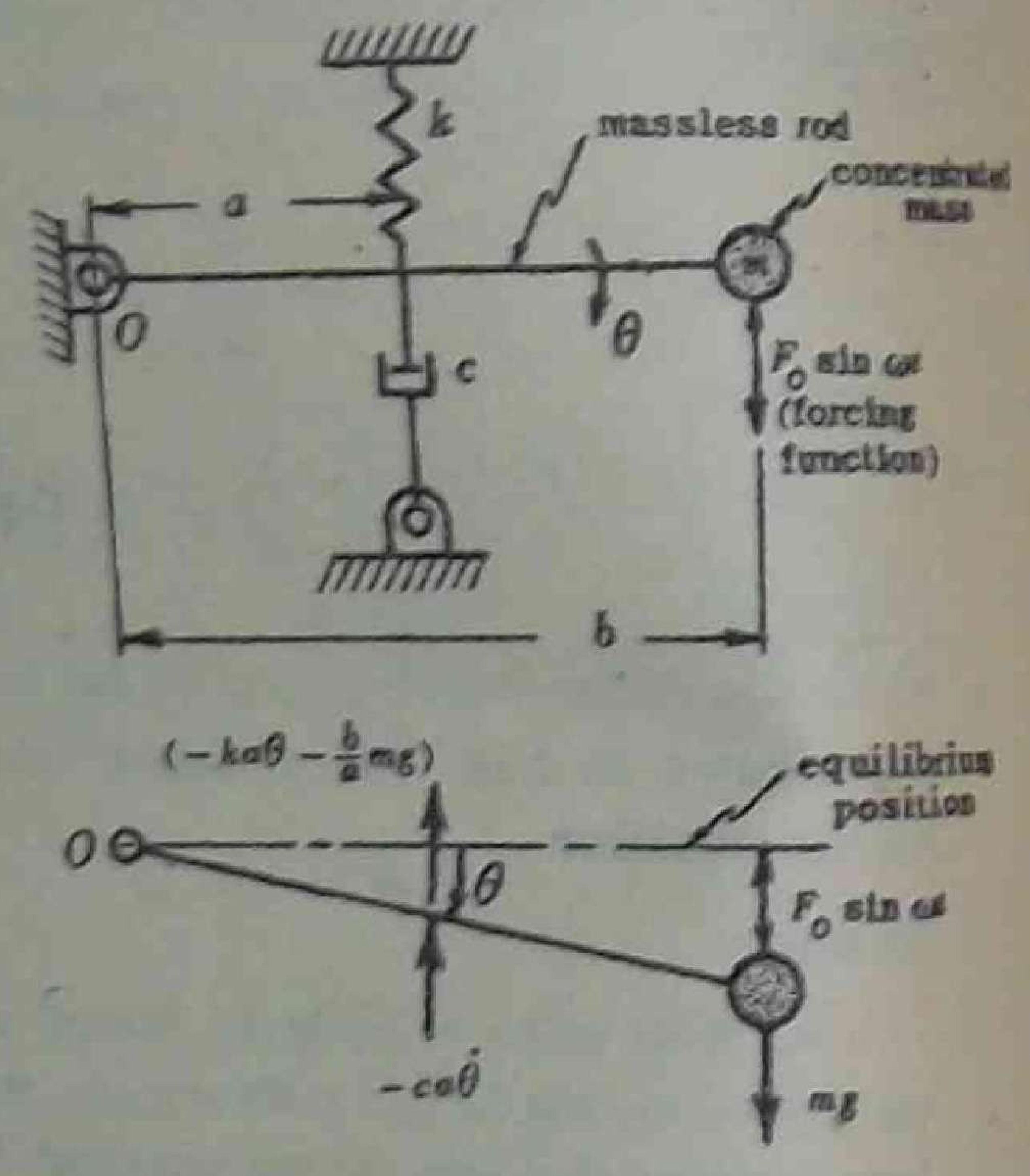


Fig. 7-6

5. A motor is mounted on springs. A small unbalance of the rotor will cause vibration when the motor is operating. Analyze this situation in order that we may be able to decide upon suitable spring characteristics for the mounting. Consider vertical movement only. Refer to Fig. 7-7.

Solution:

We adopt the following symbols:

M = total mass of motor.

me = unbalance of rotor (product of unbalanced mass and radius).

k = spring constant (effect of all springs acting together).

c = damping constant to account for friction (mostly internal frictional effects between parts and within materials, a small value).

ω = motor speed, radians per unit time.

ωt = rotation angle of the unbalanced mass, measured from the horizontal.

x = vertical displacement of the motor, measured from the position of static equilibrium.

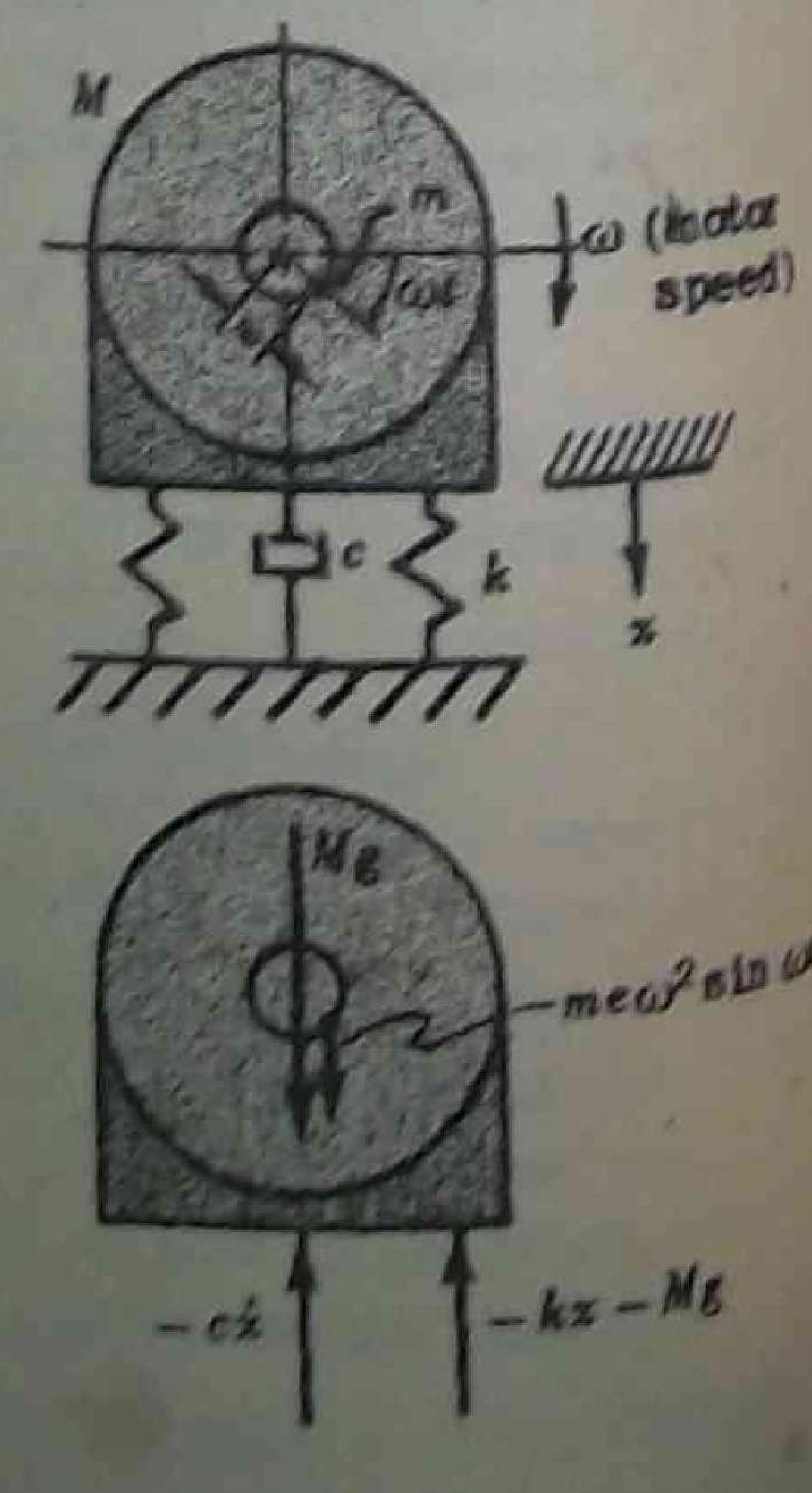


Fig. 7-7

The motor as a whole has vertical acceleration $\ddot{x}$. In addition, the unbalanced mass m has the acceleration $-e\omega^2 \sin \omega t$ in the vertical direction. The external forces are the spring and damping forces plus the weight Mg. Then

$$-c\dot{x} - kx - Mg + Mg = M\ddot{x} - me\omega^2 \sin \omega t$$

or

$$M\ddot{x} + c\dot{x} + kx = +me\omega^2 \sin \omega t$$

(Note: Whether the term on the right hand side of the equation comes out with a positive or negative sign, and whether it be $me\omega^2 \sin \omega t$ or $me\omega^2 \cos \omega t$ depends on the reference for the rotation angle ωt and the sense of rotation ω, as well as the assumed positive sense for displacement x. If, for example, we had chosen to measure ωt clockwise from the positive vertical axis, the forcing function would have been $me\omega^2 \cos \omega t$. This would not change final results of the analysis.)

The above differential equation is of the same form as that discussed earlier for the general case. However, the amplitude of the forcing function, instead of being a simple constant F_o, is a function of ω. We could apply the results listed earlier, but instead will work out the details.

Assume a steady state solution of the form $x = Y \sin (\omega t - \phi)$. Then

$$\dot{x} = Y\omega \cos (\omega t - \phi) \quad \text{and} \quad \ddot{x} = -Y\omega^2 \sin (\omega t - \phi)$$

Substituting into the differential equation, we have

$$M[-Y\omega^2 \sin (\omega t - \phi)] + cY\omega \cos (\omega t - \phi) + kY \sin (\omega t - \phi) = me\omega^2 \sin \omega t$$

or

$$-MY\omega^2 (\sin \omega t \cos \phi - \cos \omega t \sin \phi)$$
$$+ cY\omega(\cos \omega t \cos \phi + \sin \omega t \sin \phi)$$
$$+ kY (\sin \omega t \cos \phi - \cos \omega t \sin \phi) = me\omega^2 \sin \omega t$$

Equating coefficients of $\sin \omega t$, $\quad -MY\omega^2 \cos \phi + cY\omega \sin \phi + kY \cos \phi = me\omega^2$.

Equating coefficients of $\cos \omega t$, $\quad MY\omega^2 \sin \phi + cY\omega \cos \phi - kY \sin \phi = 0$.

Simultaneous solution of the last two equations written yields

$$Y = \frac{me\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}, \quad \cos \phi = \frac{k - M\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}, \quad \sin \phi = \frac{c\omega}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}$$

Hence the steady state solution of the differential equation is

$$x = \frac{me\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \sin (\omega t - \phi)$$

We now investigate the force transmitted to the base. This will be the sum of the spring and damping forces.

$$kx + c\dot{x} \quad \text{or} \quad kY \sin(\omega t - \phi) + cY\omega \cos (\omega t - \phi)$$

which can be put in the form $\quad Y\sqrt{k^2 + (c\omega)^2} \sin(\omega t - \phi + \beta)$

where $(-\phi + \beta)$ is the phase angle between the exciting force $me\omega^2 \sin \omega t$ and the transmitted force.

The important thing for our purposes is the amplitude F_{TR} of the transmitted force:

$$F_{TR} = Y\sqrt{k^2 + (c\omega)^2} = \frac{me\omega^2 \sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}$$

A better understanding of this is obtained if we put it into dimensionless form as follows.

Let $r = \frac{\omega}{\omega_n} = \frac{\omega}{\sqrt{k/M}}$ and $\xi = \frac{c}{c_c} = \frac{c}{2\sqrt{kM}}$, where ω_n = natural frequency and c_c = critical damping factor. Then

$$\frac{F_{TR}}{me\omega^2} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

The ratio of the amplitude of the transmitted force to the amplitude of the forcing function is called **transmissibility ratio**.

The original question posed in this problem was that of deciding upon suitable spring characteristics. We wish the transmitted force to be small in comparison to the force which would be transmitted if the motor frame were bolted directly to the base. This means we wish the quantity

$$\text{T.R.} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

to be small compared to unity. Friction will be small, unless we deliberately build a damping device into the mounting. Let us estimate $\xi = 0.05$ and solve for r needed to make T.R. = 0.1 :

$$0.1 = \frac{\sqrt{1 + 4(.05r)^2}}{\sqrt{(1-r^2)^2 + 4(.05r)^2}} \qquad \text{from which } r = 3.40.$$

Note: If we had assumed zero friction the result would have been $r = 3.41$, so for quick estimates in the kind of situation described here we might as well ignore friction.

If r is to be 3.40, ω_n must be $\omega/3.40$; thus $\sqrt{k/M} = \omega/3.40$ or $k = M\omega^2/11.56$.

Now let us suppose that the motor weighs 42 lb and operates at 1150 rpm. Then k must be

$$k = \frac{M\omega^2}{11.56} = \frac{(42/32.2)(2\pi \times 1150/60)^2}{11.56} = 1635 \text{ lb/ft} = 136 \text{ lb/in}$$

If we use 4 springs in parallel, each spring constant should be $136/4 = 34$ lb/in.

6. Part of a processing operation requires a screening table to be reciprocated with an amplitude of 0.025 in. at a frequency of 6 cycles per sec. The table is to have two spring steel supports, as shown in Fig. 7-8, each with a spring constant k defined as the force on the upper end of a steel support divided by the corresponding deflection at that point. Weight of table will be approximately 80 lb. A solenoid with a sinusoidal force output, $F_o \sin \omega t$, is to be used to drive the device. For what spring constant k should the supports be designed? If effective friction is estimated to be equivalent to $c = 0.05\,c_c$, what peak force F_o must the solenoid provide?

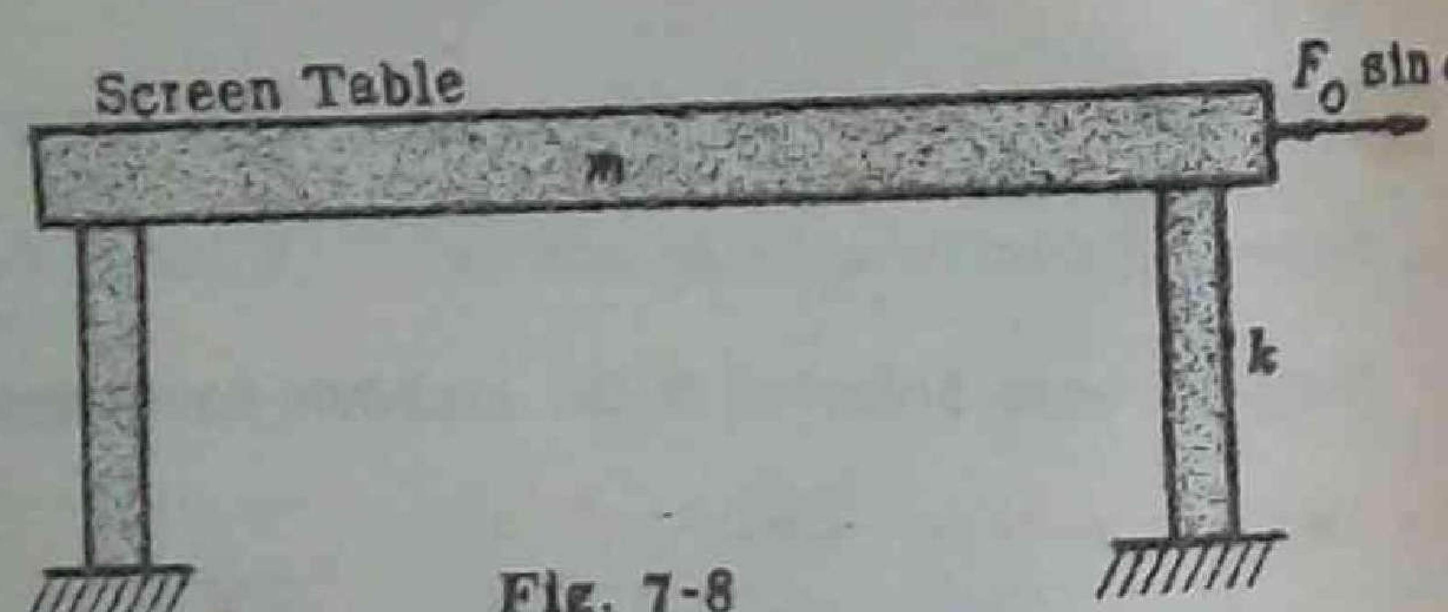


Fig. 7-8

Solution:

This is a forced, steady state vibration situation. The amplitude of vibration is

$$Y = \frac{F_o}{\sqrt{(k_e - m_e\omega^2)^2 + (c_e\omega)^2}}$$

where $m_e = m$, the mass of the table,
$k_e = 2k$ (since there are two springs each having spring constant k),
$c_e = c = 0.05\,c_c = (0.05)(2)\sqrt{k_e m}$,
$\omega = (6)(2\pi) = 12\pi$ rad/sec.
$Y = 0.025$ in., the desired amplitude.

Examination of the above equation for Y shows that Y is near maximum, for a given F_o, at resonance, i.e. when ω equals the natural frequency of the system. Hence we should design so that

$$k_e = 2k = m\omega^2 \quad \text{or} \quad k = \tfrac{1}{2}m\omega^2 = \tfrac{1}{2}(80/32.2)(12\pi)^2 = 1765 \text{ lb/ft} = 147 \text{ lb/in}$$

At resonance, $Y = F_o/c\omega$. Then the peak solenoid force required is

$$F_o = c\omega Y = 0.05\,c_c\omega Y = (0.05)(2\sqrt{k_e m})\omega Y = (0.05)\left(2\sqrt{(2)(1765)(80/32.2)}\right)(12\pi)(0.025/12)$$
$$= 0.74 \text{ lb}$$

7. It is proposed to mount a spin dryer basket as shown in Fig. 7-9. Suitable spring and damper characteristics are to be selected for the following conditions.

Total weight of basket plus contents = 50 lb
Rotational speed = 400 rpm
Maximum unbalance assumed = 20 lb-in
(product of weight and eccentricity)
Amplitude of vibration in any direction to be not more than $\frac{1}{2}$ in. at resonance.

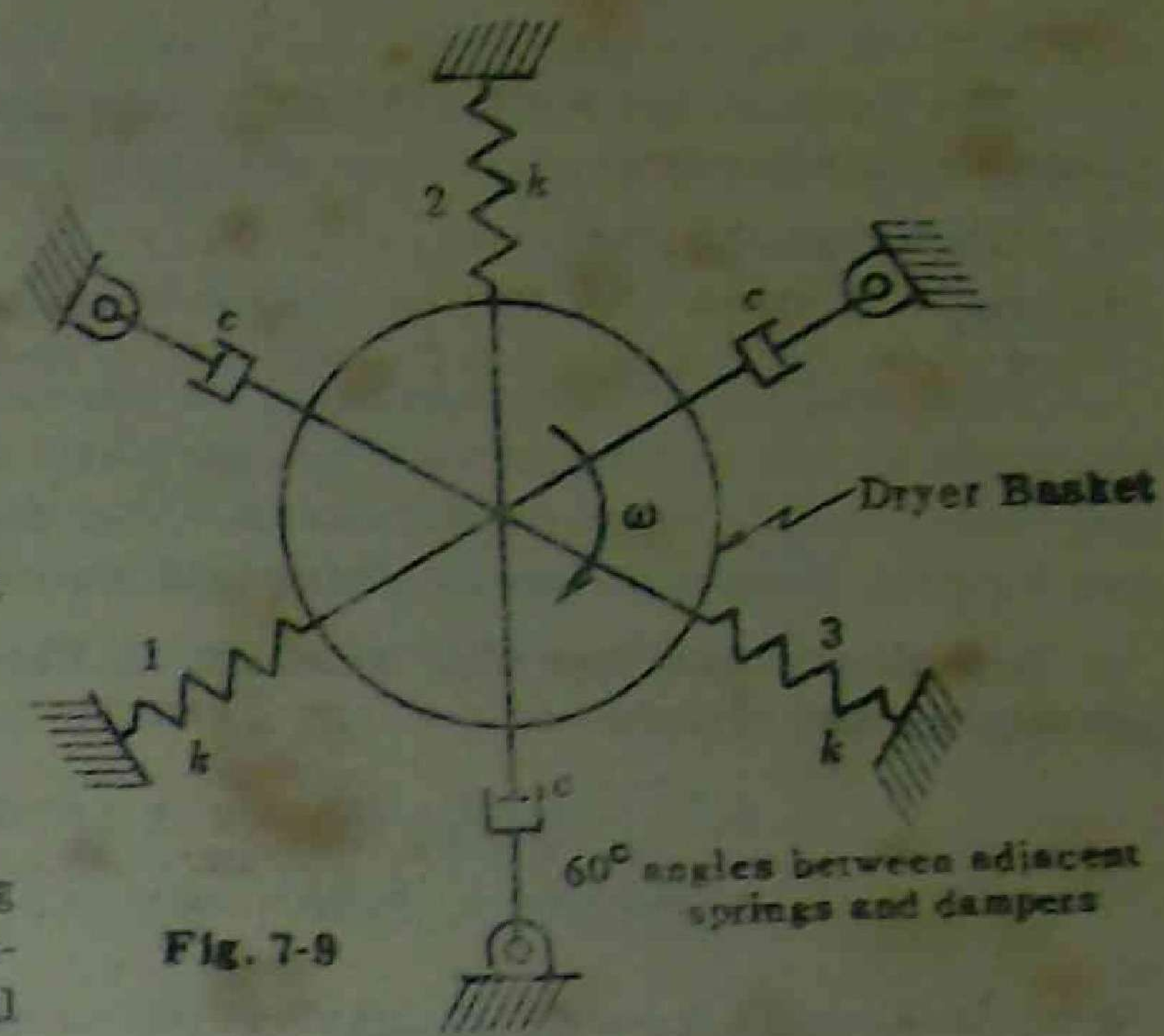


Fig. 7-9

Solution:

Choose X and Y coordinates as shown in Fig. 7-10.

Consider a small deflection x of the basket center. Spring 1 will stretch, spring 3 will compress, and spring 2 will undergo a negligible change of length. The spring forces will be approximately as indicated in Fig. 7-11.

The net spring force in the X direction is

$$F_s = -2\cos 30^\circ\, kx \cos 30^\circ = -1.5kx$$

In other words, the effective spring constant in the X direction is $1.5k$. A similar analysis would yield the same value for the effective spring constant in the Y direction.

If damping forces in the X and Y directions were investigated in the same fashion as above for spring forces, we would find that the effective damping factor in both X and Y directions is $1.5c$.

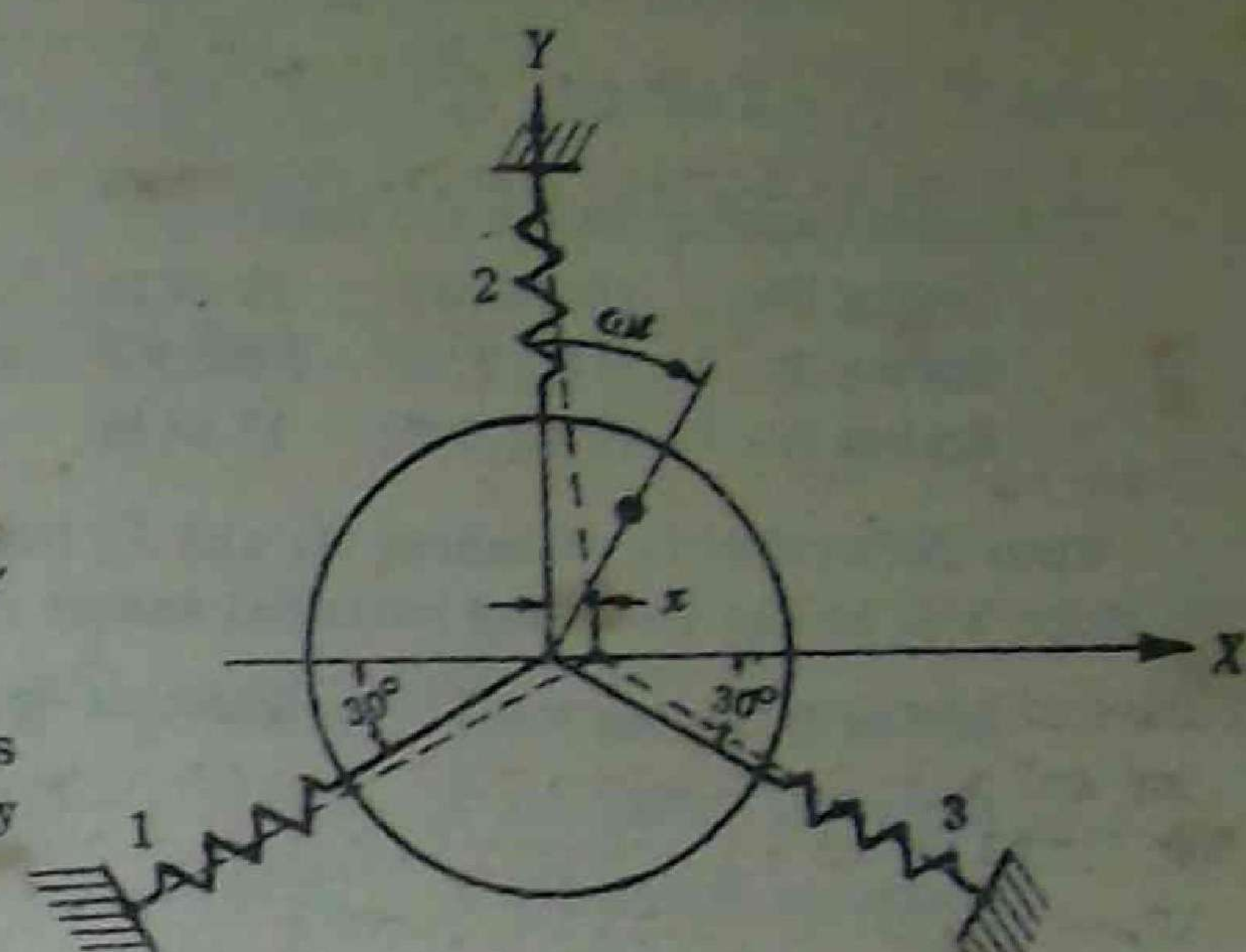


Fig. 7-10

Because all coefficients in the differential equations for X and Y motions will be alike, we need investigate only one equation.

$$M\ddot{x} + 1.5c\dot{x} + 1.5kx = (me)\omega^2 \sin \omega t$$

The displacement amplitude will be

$$Y = \frac{me\omega^2}{\sqrt{(1.5k - M\omega^2)^2 + (1.5c\omega)^2}}$$

by analogy with Prob. 5, for which the differential equation was identical in form; and the amplitude of the transmitted force will be

$$F_{TR} = \frac{me\omega^2\sqrt{(1.5k)^2 + (1.5c\omega)^2}}{\sqrt{(1.5k - M\omega^2)^2 + (1.5c\omega)^2}}$$

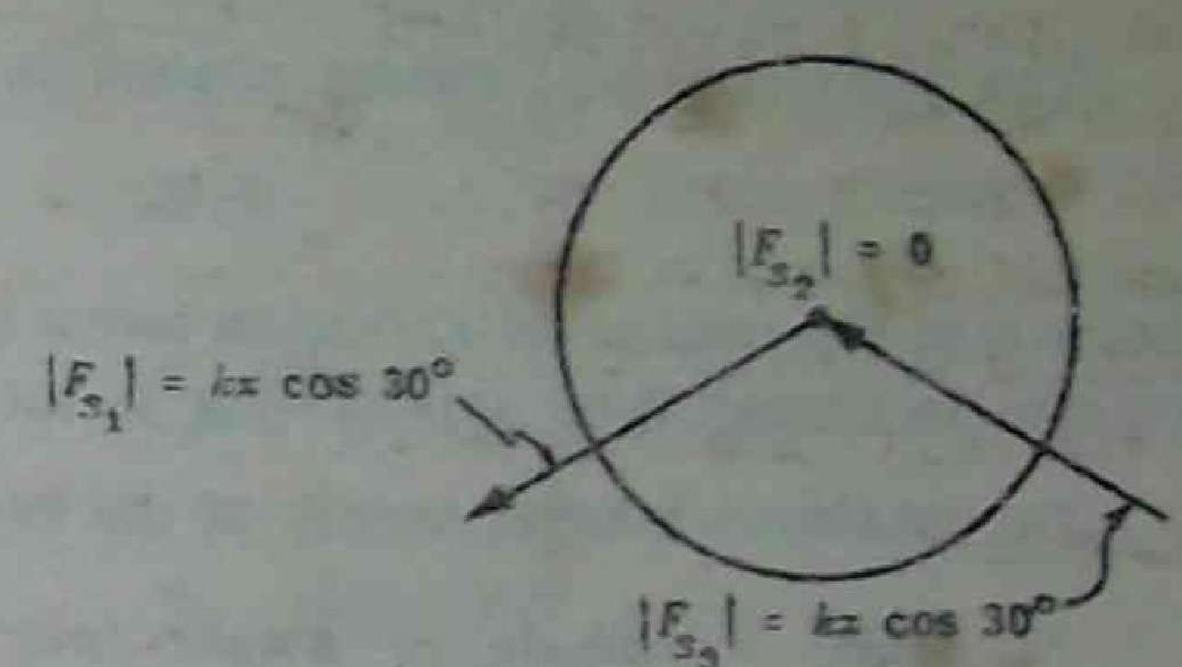


Fig. 7-11

We saw in Prob. 5 that, to keep the transmitted force small, we make the natural frequency low compared to the operating frequency which is specified. For a tentative design we shall choose to make $\omega/\omega_n = 3$. Since in this system the natural frequency is $\omega_n = \sqrt{1.5k/M}$, this means that we shall design so that $1.5k = \omega_n^2 M = (\omega/3)^2 M$ or

$$k = \frac{\omega^2 M}{9(1.5)} = \frac{(400 \times 2\pi/60)^2(50/32.2)}{9(1.5)} = 201 \text{ lb/ft} = 16.8 \text{ lb/in}$$

We now calculate the damping factor c required to limit the displacement amplitude to $\frac{1}{2}$ in. at resonance. At resonance,

$$Y = \frac{me\omega_n^2}{\sqrt{0 + (1.5c\omega_n)^2}} \quad \text{or} \quad c = \frac{me\omega_n}{1.5Y} = \frac{0.0517(13.9)}{1.5(1/24)} = 11.5 \text{ lb-sec/ft} = 0.96 \text{ lb-sec/in}$$

where $Y = 1/24$ ft, $me = (20/32.2)(1/12) = 0.0517$ slug-ft, $\omega_n = (2\pi \times 400/60)/3 = 13.9$ rad/sec.

Answer. Design for $\omega_n = \omega/3$: $k = 16.8$ lb/in, $c = 0.96$ lb-sec/in.

8. In Fig. 7-12, m_1 weighs 10 lb, m_2 weighs 20 lb, $k_1 = 8$ lb/in, $k_2 = 10$ lb/in, and $k_3 = 5$ lb/in. Using the energy method, determine the natural frequency of vibration in the first mode, for vertical motion only.

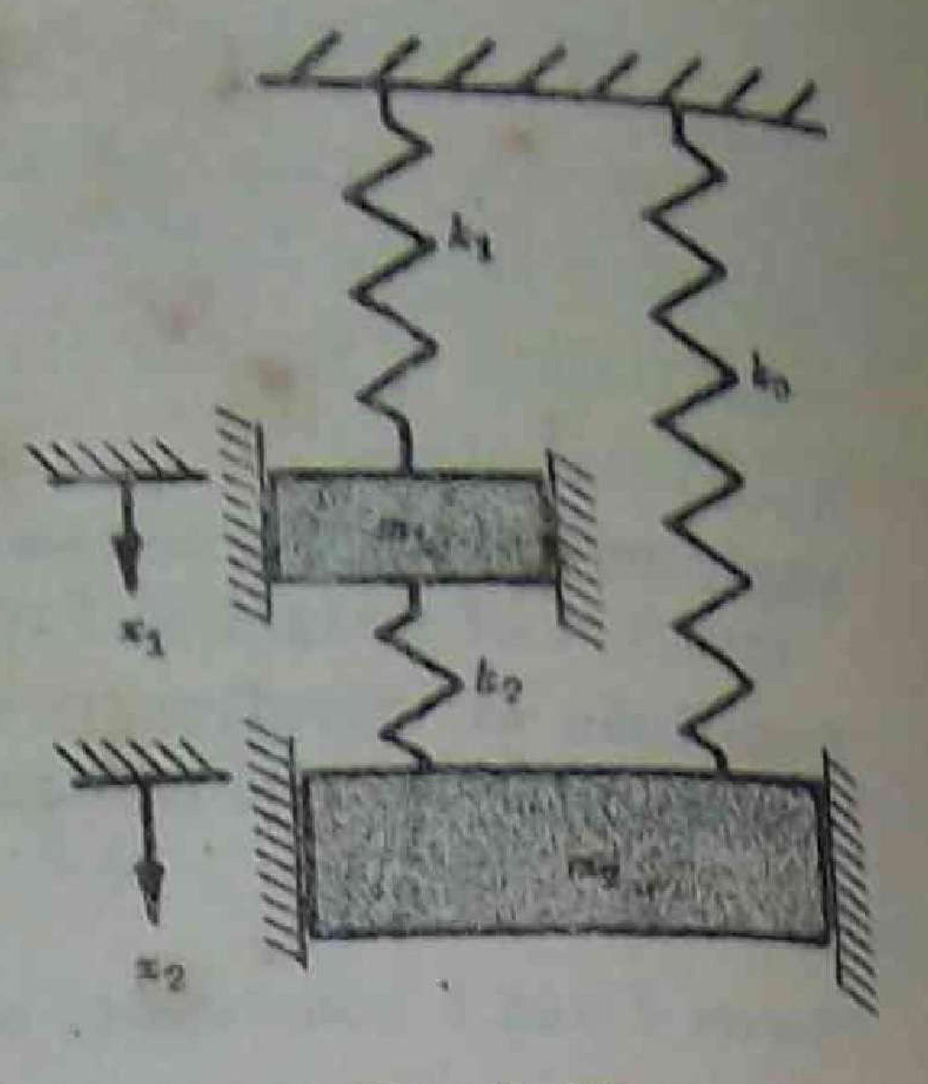


Fig. 7-12

Solution:

Let X_1 and X_2 be the amplitudes of the absolute displacements of masses m_1 and m_2 respectively from the static equilibrium position. Let δ_1 and δ_2 be the displacements of the masses under their own weights, measured from the unstressed spring positions.

Referring to the free body sketches in Fig. 7-13, for static equilibrium we have

$$-k_1\delta_1 + k_2(\delta_2 - \delta_1) + m_1 g = 0$$
$$-k_2(\delta_2 - \delta_1) - k_3\delta_2 + m_2 g = 0$$

or

$$-8\delta_1 + 10(\delta_2 - \delta_1) + 10 = 0$$
$$-10(\delta_2 - \delta_1) - 5\delta_2 + 20 = 0$$

from which $\delta_1 = 2.060$ in., $\delta_2 = 2.708$ in.

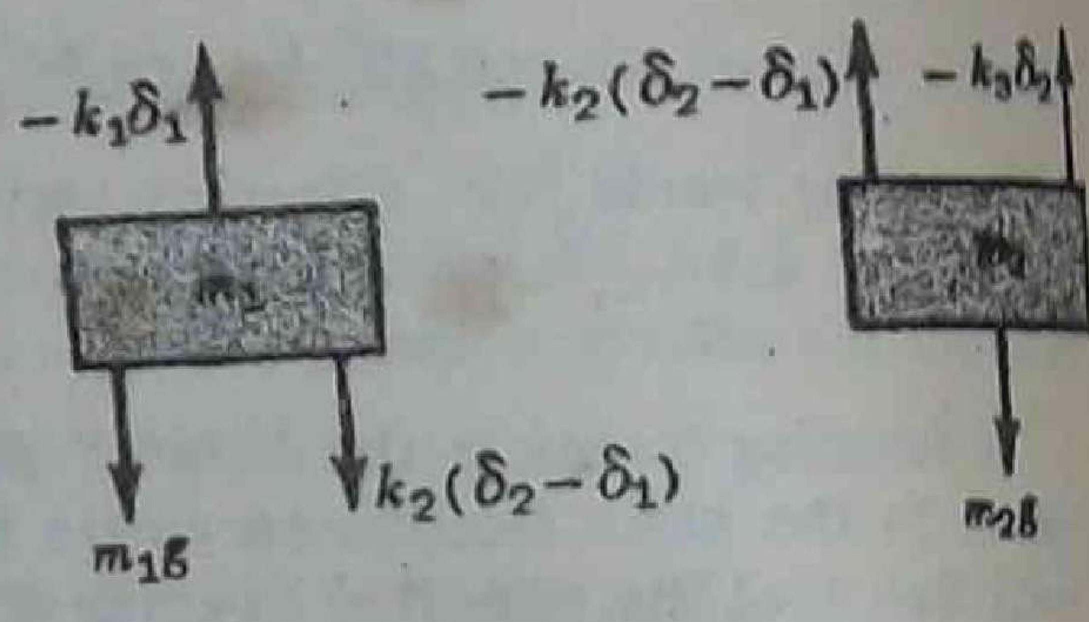


Fig. 7-13

The initial spring forces are then:

Spring No. 1, $(8)(2.060) = 16.48$ lb
Spring No. 2, $(10)(2.708 - 2.060) = 6.48$ lb
Spring No. 3, $(5)(2.708) = 13.54$ lb.

Upon deflection to distances X_1 and X_2 from static equilibrium there will be the following potential energy changes:

Stored in Spring No. 1, $(16.48)X_1 + \frac{1}{2}(8)X_1^2$
Stored in Spring No. 2, $(6.48)(X_2 - X_1) + \frac{1}{2}(10)(X_2 - X_1)^2$
Stored in Spring No. 3, $(13.54)X_2 + \frac{1}{2}(5)X_2^2$
Change of elevation, m_1, $-10X_1$
Change of elevation, m_2, $-20X_2$.

The total potential energy change in moving from the static equilibrium position is then

$$\text{P.E.} = 4X_1^2 + 5(X_2 - X_1)^2 + 2.5X_2^2$$

Notice that the change-of-elevation terms just cancel the initial-spring-force terms. We might have treated the systems as though they were moving in the horizontal plane, without affecting the results.

The maximum kinetic energy of the moving masses, assuming sinusoidal motion at frequency ω, will be

$$\text{K.E.} = \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2 = \tfrac{1}{2}(10/g)(X_1\omega)^2 + \tfrac{1}{2}(20/g)(X_2\omega)^2 = (5X_1^2 + 10X_2^2)\omega^2/g$$

Equating K.E. and P.E., we obtain $\omega^2 = \dfrac{[4X_1^2 + 5(X_2 - X_1)^2 + 2.5X_2^2]g}{5X_1^2 + 10X_2^2}$ which can be put in the form

$$\omega^2 = \frac{[4 + 5(X_2/X_1 - 1)^2 + 2.5(X_2/X_1)^2]g}{5 + 10(X_2/X_1)^2}$$

The final step is to assume values for the ratio X_2/X_1 and calculate ω. The lowest resulting value for ω is the most nearly correct. (Note: $g = 386$ in/sec^2)

Assumed X_2/X_1	Calculated ω^2	ω
1.6	$0.397\,g$	12.38 rad/sec
1.4	$0.394\,g$	12.33 rad/sec
1.2	$0.402\,g$	12.45 rad/sec

The answer is very nearly 12.33 rad/sec. Notice that the result is not very sensitive to the assumed ratio of X_2/X_1. Usually, a good value to try first is the ratio of static deflections. In this case, $\delta_2/\delta_1 = 2.708/2.060 = 1.31$. If we had chosen $X_2/X_1 = 1.31$, we would have been as close to the final value of $\omega = 12.33$ rad/sec as our sliderule can take us.

Critical Speeds of Shafts

ALL ROTATING SHAFTS, even in the absence of external load, deflect during rotation. The magnitude of the deflection depends upon the stiffness of the shaft and its supports, the total mass of shaft and attached parts, the unbalance of the mass with respect to the axis of rotation, and the amount of damping in the system. The deflection, considered as a function of speed, shows maximum values at so-called critical speeds. For any shaft there are an infinite number of critical speeds, but only the lowest (first) and occasionally the second are generally of interest to the designer. The others will usually be so high as to be well out of the range of operating speeds.

AT THE FIRST CRITICAL SPEED the shaft will bend to the simplest shape possible. At the second critical speed it will bend to the second simplest shape possible, etc. For example, a shaft supported at its ends and having two relatively large (compared to shaft) masses attached, will bend to the configurations shown in Fig. 8-1(a) and Fig. 8-1(b) at the first and second critical speeds, respectively.

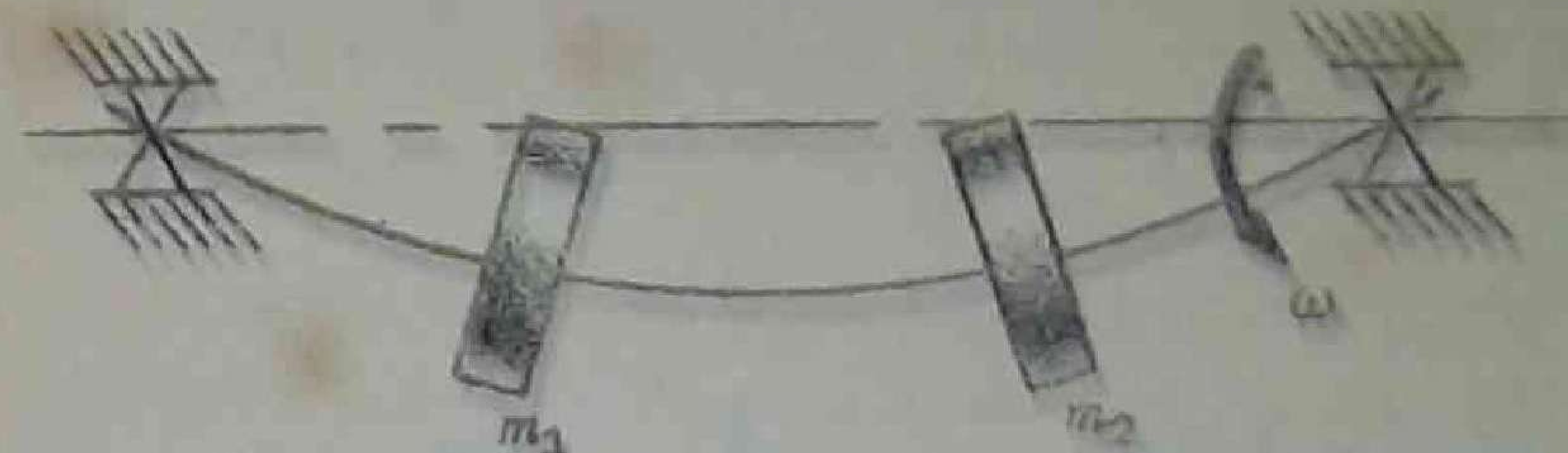
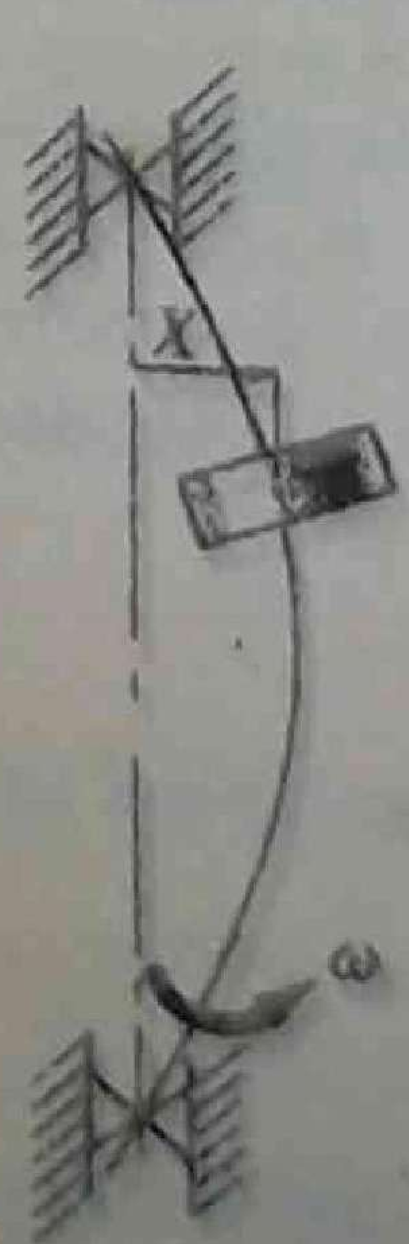


Fig. 8-1 (a)

Fig. 8-1 (b)

THE NATURAL FREQUENCY of the shaft in bending is very nearly the same as the critical speed, and is usually taken as being the same thing. There is a difference, usually quite small, due to the gyroscopic action of the masses.



FOR A SHAFT WITH SINGLE ATTACHED MASS (Fig. 8-2 and 8-3), if the shaft mass is small compared to the attached mass, the first critical speed can be calculated approximately as

$$\omega_c = \sqrt{k/m} \quad \text{rad/unit time}$$

Fig. 8-2(a)

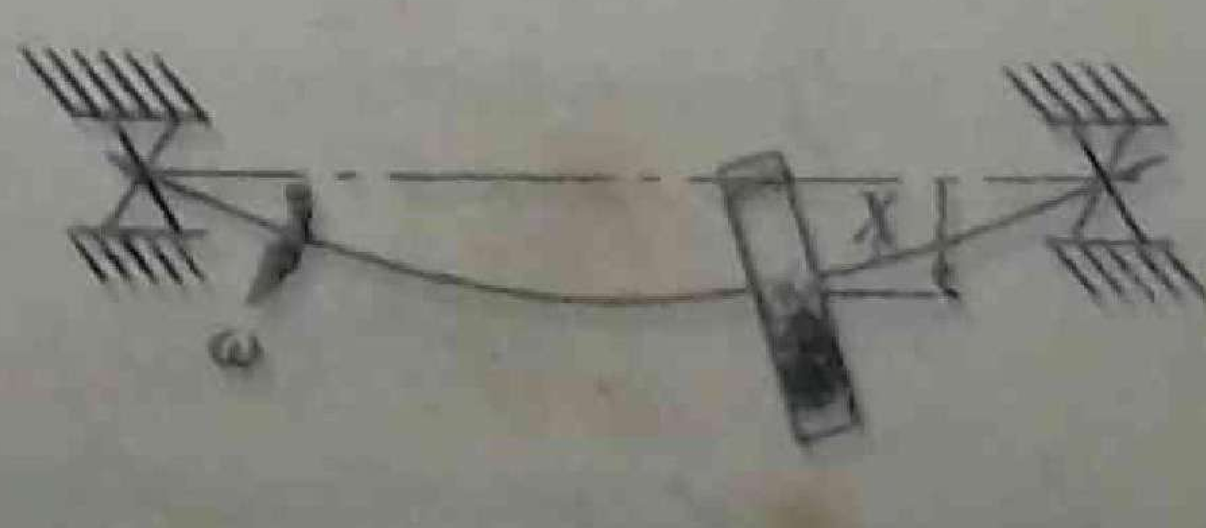


Fig. 8-2(b)

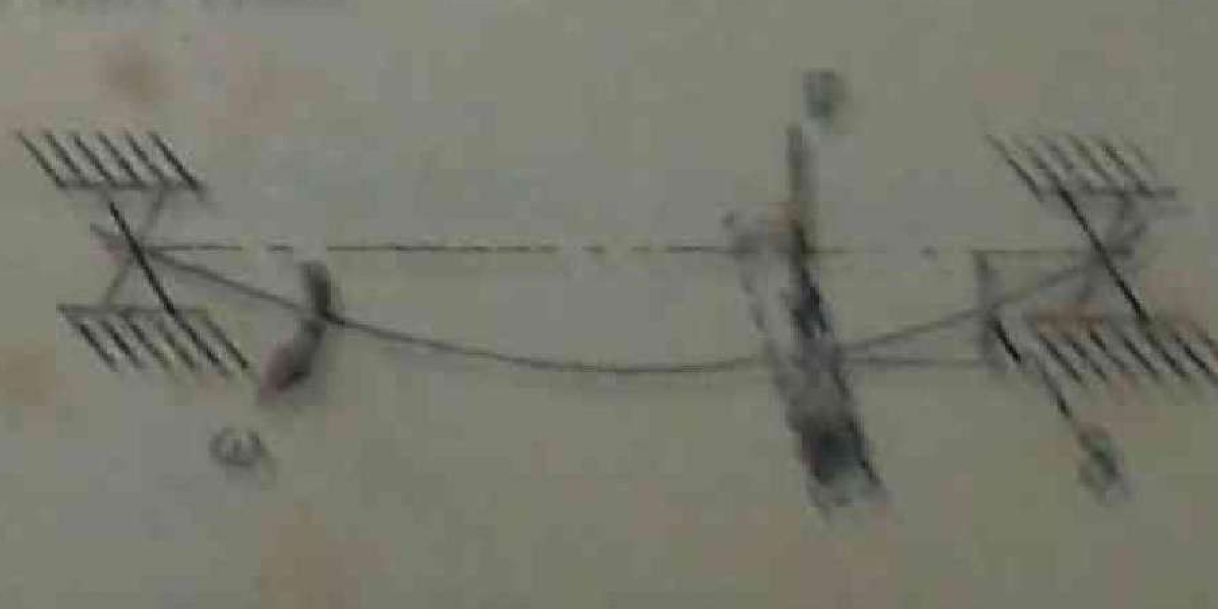


Fig. 8-3

where m is the mass and k is the shaft spring constant (force required for one unit deflection at the mass location). This relation is independent of shaft inclination (horizontal, vertical, or intermediate). Symbol X in Fig. 8-2 represents the shaft deflection, during rotation, at the mass location. Also

$$\omega_c = \sqrt{g/\delta} \quad \text{rad/unit time}$$

where δ is the static deflection (deflection, at the mass location, which would be caused by a force $mg = W$), and g is the gravitational constant (32.2 ft/sec² or 386 in/sec²).

FOR A SHAFT OF CONSTANT CROSS SECTION, simply supported at the ends, with no mass involved other than that of the shaft itself, the first critical speed is very nearly

$$\omega_c = \sqrt{\frac{5}{4}\left(\frac{g}{\delta(\max)}\right)} \quad \text{rad/unit time}$$

where $\delta(\max)$ is the maximum static deflection caused by a uniformly distributed load equal to the weight of the shaft.

FOR A SHAFT OF NEGLIGIBLE MASS CARRYING SEVERAL CONCENTRATED MASSES, (see Fig. 8-4), the first critical speed is approximately

$$\omega_c = \sqrt{\frac{g \sum_1^j W_n \delta_n}{\sum_1^j W_n \delta_n^2}} \qquad \text{Rayleigh-Ritz Equation}$$

where W_n = weight of n^{th} mass,
δ_n = static deflection at the n^{th} mass,
j = total number of masses.

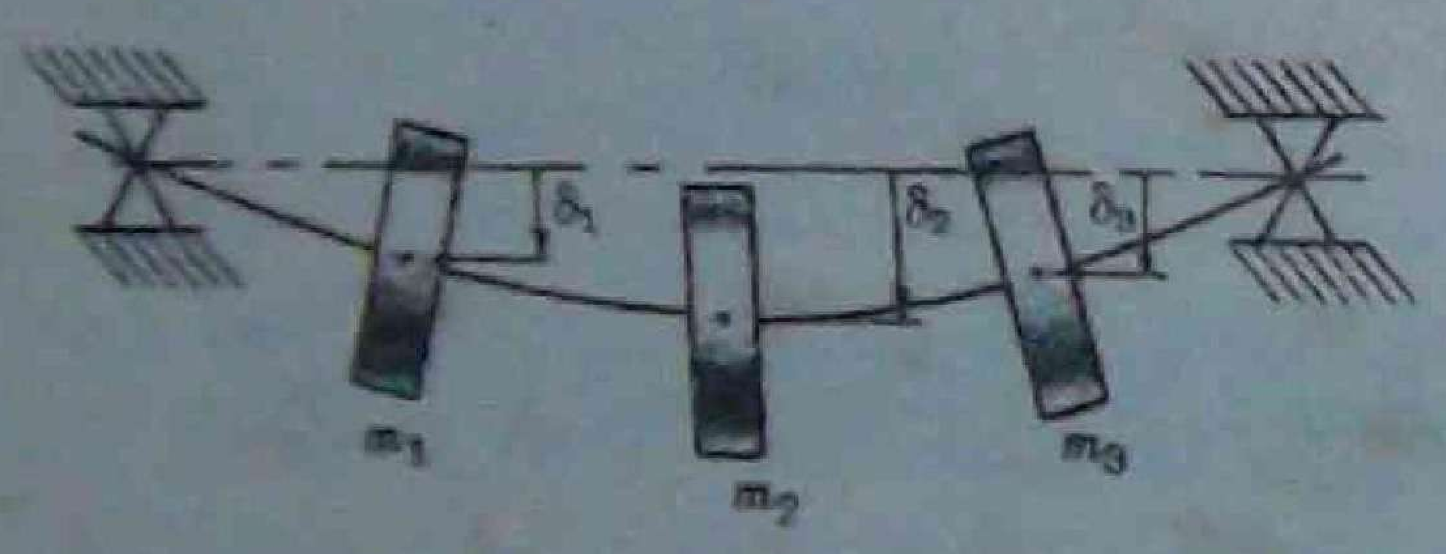


Fig. 8-4

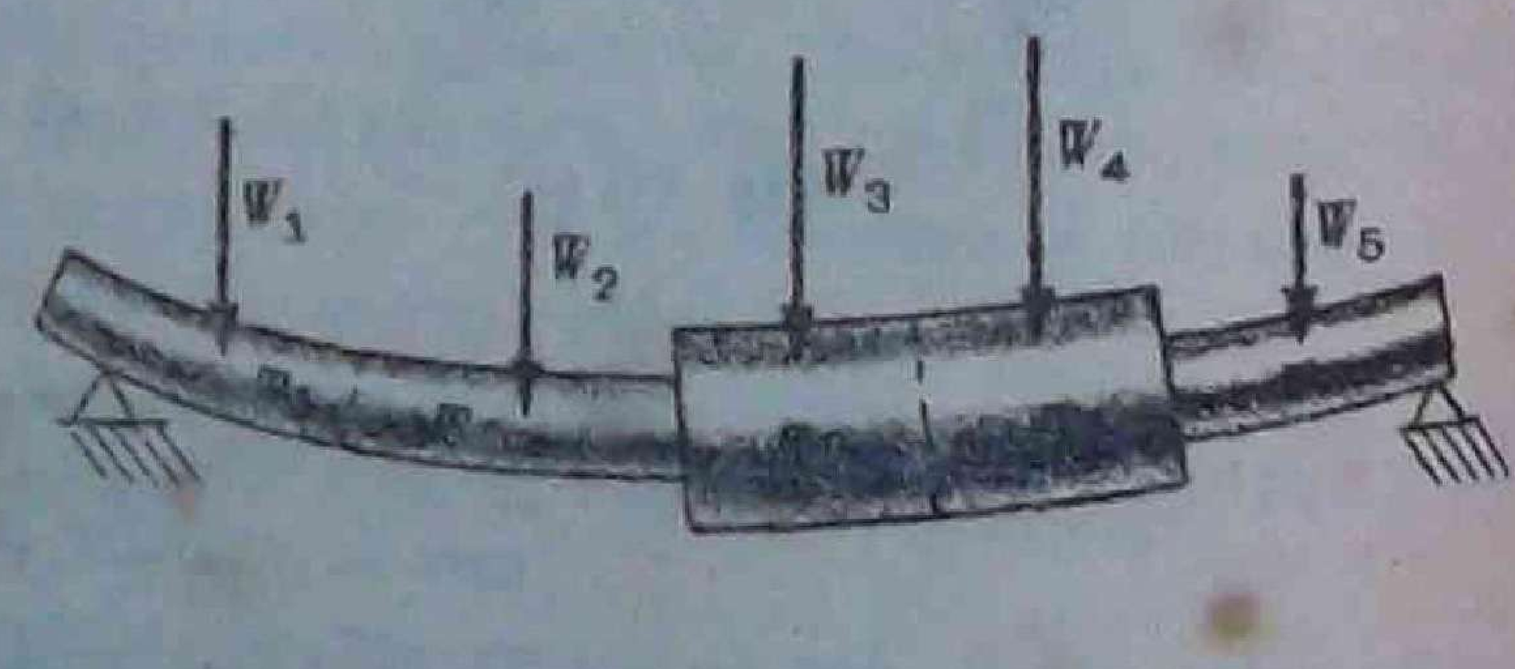


Fig. 8-5

This same equation can be used for estimating the first critical speed of a shaft with distributed mass. Refer to Fig. 8-5 above. Break the distributed mass into a number of pieces, m_1, m_2, m_3, etc. Treat the mass of each piece as though concentrated at its center of gravity. The number of subdivisions to use is a matter to be learned by experience, but it will be found that good accuracy is obtained with rather crude approximations.

THE DUNKERLEY EQUATION, another approximation for the first critical speed of a multimass system, is

$$\frac{1}{\omega_c^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} + \cdots \qquad \text{Dunkerley Equation}$$

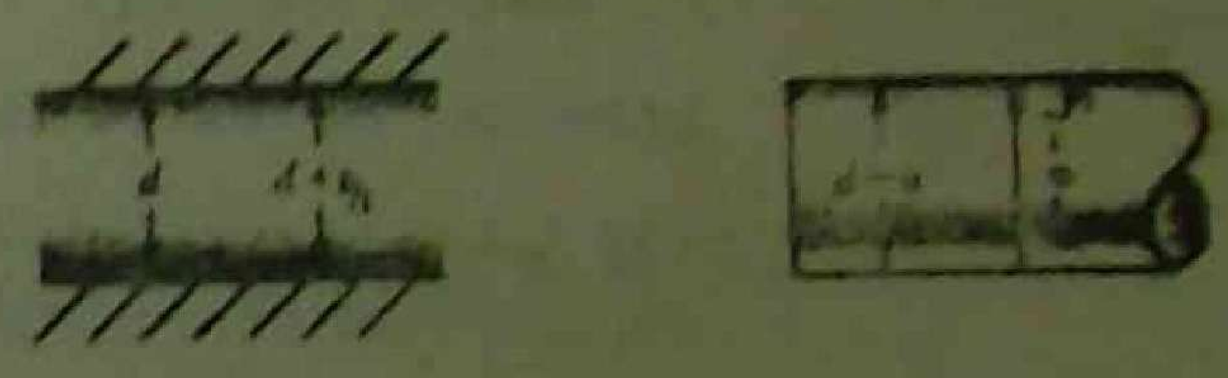
where ω_c is the first critical speed of the multimass system, ω_1 is the critical speed which would exist if only mass No. 1 were present, ω_2 the critical speed with only mass No. 2, etc.

It is important to keep in mind that both the Rayleigh-Ritz and the Dunkerley equations are approximations to the first natural frequency of vibration, which is assumed to be nearly equal to the critical speed of rotation. In general, the Rayleigh-Ritz equation overestimates and the Dunkerley equation underestimates the natural frequency.

Class of Fit	Method of Assembly	Allowance	Average Interference (negative allowance)	Hole Tolerance	Shaft Tolerance
1. Loose	Interchangeable	$0.0025\,d^{\frac{2}{3}}$	--------	$0.0025\,d^{\frac{1}{3}}$	$0.0025\,d^{\frac{1}{3}}$
2. Free	"	$0.0014\,d^{\frac{2}{3}}$	--------	$0.0013\,d^{\frac{1}{3}}$	$0.0013\,d^{\frac{1}{3}}$
3. Medium	"	$0.0009\,d^{\frac{2}{3}}$	--------	$0.0008\,d^{\frac{1}{3}}$	$0.0008\,d^{\frac{1}{3}}$
4. Snug	"	0.0000	--------	$0.0006\,d^{\frac{1}{3}}$	$0.0004\,d^{\frac{1}{3}}$
5. Wringing	Selective	--------	0.0000	$0.0006\,d^{\frac{1}{3}}$	$0.0004\,d^{\frac{1}{3}}$
6. Tight	"	--------	$0.00025\,d$	$0.0006\,d^{\frac{1}{3}}$	$0.0006\,d^{\frac{1}{3}}$
7. Medium Force	"	--------	$0.0005\,d$	$0.0006\,d^{\frac{1}{3}}$	$0.0006\,d^{\frac{1}{3}}$
8. Heavy Force or shrink	"	--------	$0.0010\,d$	$0.0006\,d^{\frac{1}{3}}$	$0.0006\,d^{\frac{1}{3}}$

ALLOWANCES AND TOLERANCES as applied to the basic hole standard are shown in Fig.3-1. Note that the hole dimensions are the same for both running and tight fits.

- d = nominal dimension
- t_h = hole tolerance
- t_s = shaft tolerance
- a = allowance
- i = selected average interference (also called negative allowance in interference fits)



For Fits (Class 1-4)

For Fits (Class 5-8)

Fig. 3-1

SELECTIVE ASSEMBLY is the practice of sorting parts into different size groups and then assembling the parts in corresponding groups, to obtain closer fits than would otherwise be economically feasible. For example, suppose 1 inch shafts are to be manufactured in accordance with a class 2 fit, with dimensions ranging from 0.9986 in. to 0.9973 in. The corresponding bearings are manufactured with dimensions ranging from 1.0000 in. to 1.0013 in. If fully interchangeable assembly is practiced, clearance would range from 0.0014 in. to 0.0040 in.

However, if it is desired to hold the clearance range from 0.0020 in. to 0.0034 in., perhaps for lubrication reasons, we might sort the shafts and bearings into two groups as follows:

Group A	Bearings	1.0000 in. to 1.0007 in.
	Shafts	0.9973 in. to 0.9980 in.
Group B	Bearings	1.0007 in. to 1.0013 in.
	Shafts	0.9980 in. to 0.9986 in.

HIGHER CRITICAL SPEEDS for multimass systems require much more extensive calculation than is necessary for the determination of the lowest (first) critical speed. Several different methods have been developed. Here we shall give the equation for a two-mass system only:

$$\frac{1}{\omega^4} - (a_{11}m_1 + a_{22}m_2)\frac{1}{\omega^2} + (a_{11}a_{22} - a_{12}a_{21})m_1m_2 = 0$$

This is a bi-quadratic equation having the positive roots $1/\omega_1$ and $1/\omega_2$, where ω_1 and ω_2 are the first and second critical speeds (or natural frequencies of vibration). The two masses are m_1 and m_2.

The constants a are influence coefficients. a_{12} is the deflection at the location of mass No. 1 that would be caused by a unit load at the location of mass No. 2, a_{11} is the deflection at No. 1 caused by unit load at No. 1, etc. Maxwell's reciprocity theorem states that $a_{12} = a_{21}$.

FOR ANY MULTIMASS SYSTEM the frequency equation is obtained by setting the following determinant equal to zero.

$$\begin{vmatrix} (a_{11}m_1 - \frac{1}{\omega^2}) & (a_{12}m_2) & (a_{13}m_3) & \cdots \\ (a_{21}m_1) & (a_{22}m_2 - \frac{1}{\omega^2}) & (a_{23}m_3) & \cdots \\ (a_{31}m_1) & (a_{32}m_2) & (a_{33}m_3 - \frac{1}{\omega^2}) & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

SOLVED PROBLEMS

1. The shaft shown in Fig. 8-6 has attached to it a gear m_1 weighing 50 lb and a flywheel m_2 weighing 100 lb. Static deflections δ_1 and δ_2 have been found to be .0012 in. and .0003 in. respectively. Determine the first critical speed, ignoring the mass of the shaft itself.

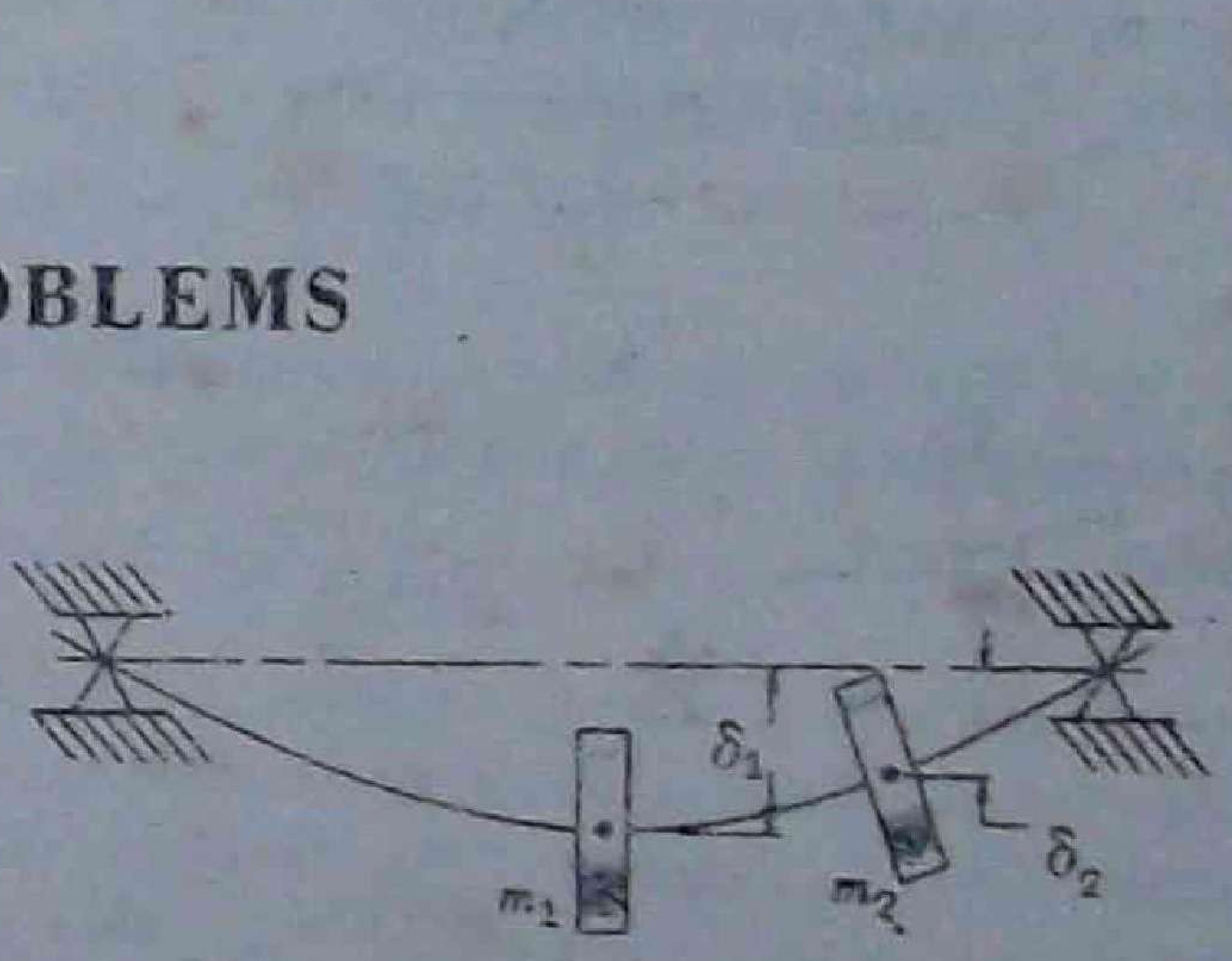


Fig. 8-6

Solution:

$$\Sigma W\delta = (50)(.0012) + (100)(.0003) = .090 \text{ lb-in}$$
$$\Sigma W\delta^2 = (50)(.0012)^2 + (100)(.0003)^2 = 81 \times 10^{-6} \text{ lb-in}^2$$

$$\omega_c = \sqrt{\frac{g\Sigma W\delta}{\Sigma W\delta^2}} = \sqrt{\frac{(386)(.090)}{81\times10^{-6}}} = 655 \text{ rad/sec} = 6250 \text{ rpm}$$

2. Derive the equation $\omega_c = \sqrt{g/\delta}$ for the critical speed of a shaft carrying a single concentrated mass. Refer to Fig. 8-7 below.

Solution:

We neglect the small tilting of the mass, ignore frictional effects, and presume some small eccentricity e of the mass center of gravity with respect to the shaft center. Then,

$$kX = m(X + e)\omega^2$$

where kX is the spring force which the shaft exerts on the mass, k being the local spring constant for the shaft, i.e. the force required at the location of m to give the shaft one unit of deflection here. $(X+e)\omega^2$ is the acceleration of the center of gravity of the mass. Solving for X, the shaft deflection at m,

$$X(k - m\omega^2) = me\omega^2 \quad \text{or} \quad X = me\omega^2/(k - m\omega^2)$$

We see that, under the assumptions made, the deflection X becomes very large when $k = m\omega^2$. Hence the critical speed is $\omega_c = \sqrt{k/m}$. But $m = W/g$; then $k/m = kg/W = g/\delta$. (By definition, the static deflection δ is the deflection which would by caused by a force equal to W; hence $W/k = \delta$.) Thus $\omega_c = \sqrt{g/\delta}$.

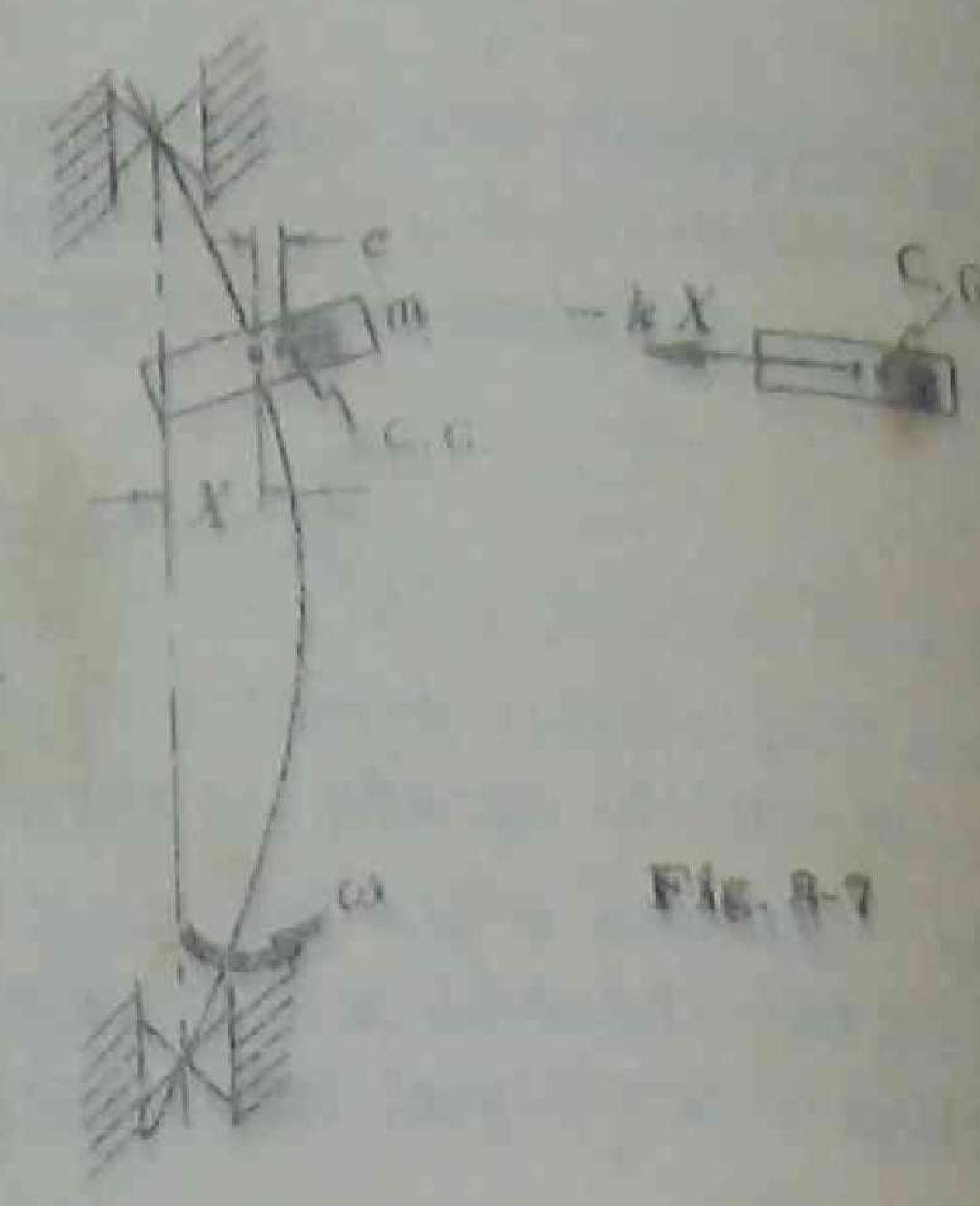


Fig. 8-7

3. Derive the equation $\omega_c = \sqrt{\dfrac{g\sum W\delta}{\sum W\delta^2}}$ for the first critical speed of a shaft with several concentrated masses. Refer to Fig. 8-8.

Solution:

We picture the shaft in free lateral vibration at the fundamental frequency ω (first mode vibration) and reason that the maximum potential energy stored in the shaft must equal the maximum kinetic energy of the moving masses.

$$\text{Max. K.E.} = \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2 + \ldots$$

The motion of the masses will be sinusoidal. Hence the maximum velocity for any mass will be $X_n\omega$, where X_n is the amplitude of displacement of that mass. Therefore,

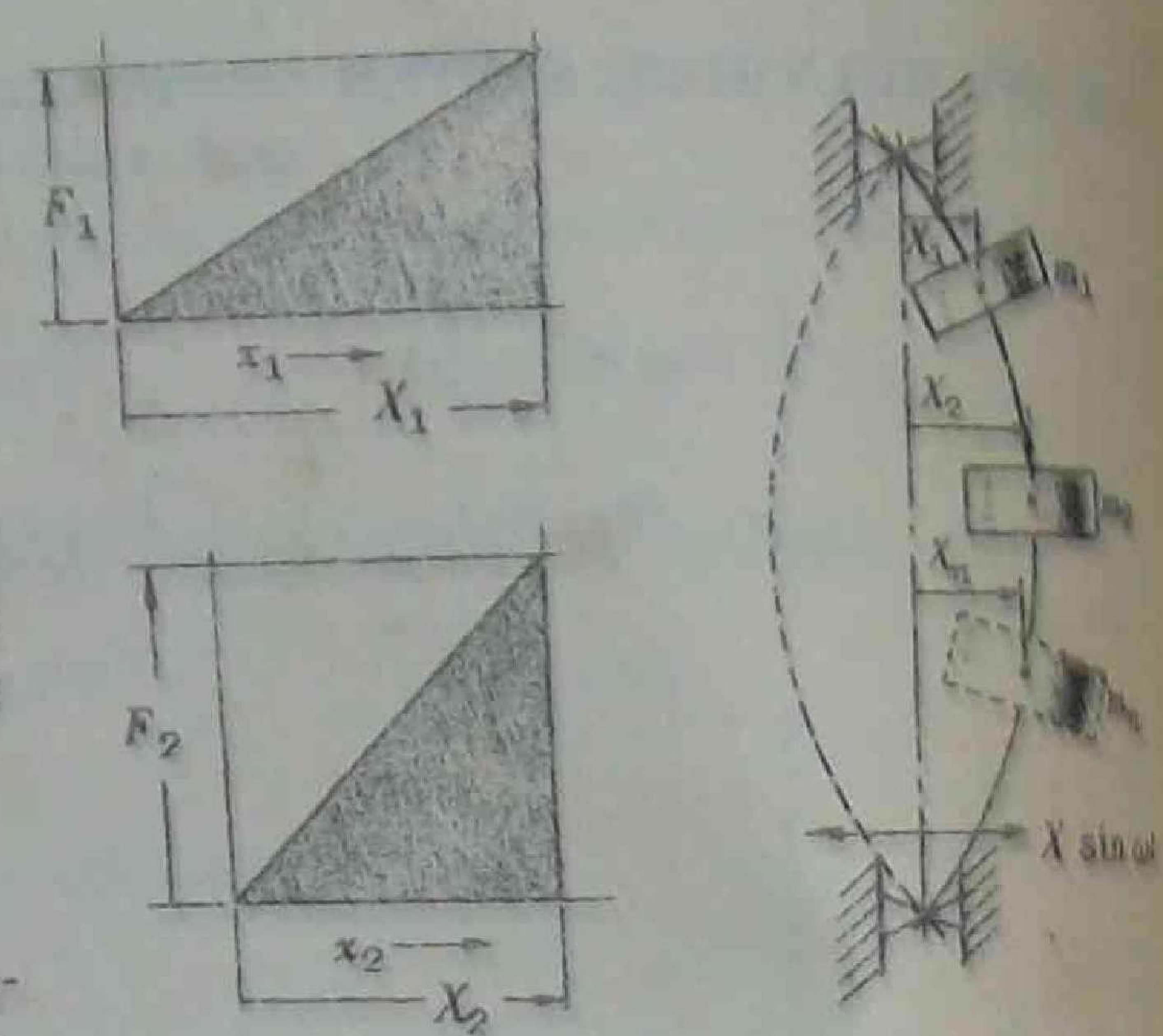


Fig. 8-8

$$\text{Max. K.E.} = \tfrac{1}{2}m_1(X_1\omega)^2 + \tfrac{1}{2}m_2(X_2\omega)^2 + \ldots = \tfrac{1}{2}\,\omega^2 \sum m_n X_n^2$$

The maximum potential energy stored in the shaft is equal to the work necessary to deflect the shaft to the shape defined by the amplitudes X_1, X_2, etc. Hence,

$$\text{Max. P.E.} = \tfrac{1}{2}k_1X_1^2 + \tfrac{1}{2}k_2X_2^2 + \ldots = \tfrac{1}{2}\sum k_n X_n^2$$

where each k is a "spring constant" whose definition can be explained as follows. Let F_1, F_2, F_3, etc., be those forces which, if acting simultaneously at locations 1, 2, 3, etc., respectively, would result in the deflections X_1, X_2, X_3, etc. Now, the shape of the shaft deflection curve depends on these forces, not on how they were built up. We might for example assume F_1 to have been applied first, then F_3, then F_2, ..., in any arbitrary fashion. We choose to assume that the forces were built up *simultaneously* from zero in a *linear* relation to the deflections at the force locations. See the force-deflection diagrams in Fig. 8-8. The work done at each force location is represented by the shaded area under the straight line of slope k.

Equating maximum kinetic and potential energies, we obtain $\omega^2 = \dfrac{\sum k_n X_n^2}{\sum m_n X_n^2}$.

We now assume that the shape to which the shaft bends during vibration is the same as the static deflection curve, i.e. we assume $X_1 = C\delta_1$, $X_2 = C\delta_2$, etc. Actually we know this to be incorrect, but it gives a reasonable approximation. Then

$$\omega^2 = \frac{\sum k_n\delta_n^2}{\sum m_n\delta_n^2} = \frac{g\sum W_n\delta_n}{\sum W_n\delta_n^2}$$

since $m_n = W_n/g$ and $k_n\delta_n = W_n$.

Assuming the natural frequency of lateral vibration ω to equal the critical speed of rotation ω_c and dropping the n subscripts for simplicity, we have finally $\omega_c = \sqrt{\dfrac{g\sum W\delta}{\sum W\delta^2}}$.

4. The two masses, m_1 and m_2, attached to the shaft of Fig. 8-9 weigh 140 lb and 60 lb respectively. Through a deflection analysis the influence coefficients for the shaft have been found to be

$$a_{11} = 2 \times 10^{-6} \text{ in/lb},$$
$$a_{22} = 12 \times 10^{-6} \text{ in/lb},$$
$$a_{12} = a_{21} = 4 \times 10^{-6} \text{ in/lb}.$$

(Remember that a_{11} is the deflection at location No. 1 caused by a 1 lb force at that location, a_{12} is the deflection at location No. 1 caused by a 1 lb force at location No. 2, etc.) Determine the first critical speed, ignoring the mass of the shaft.

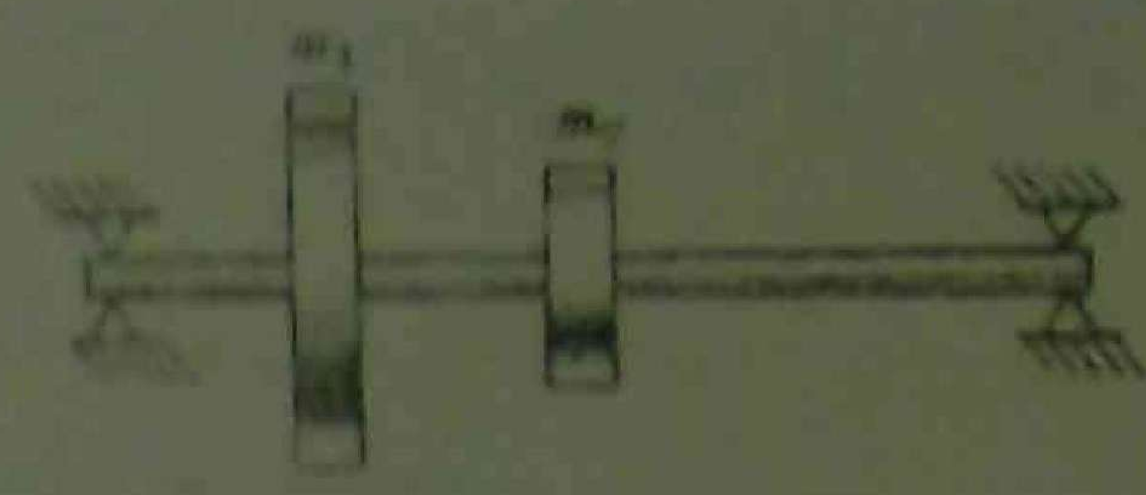


Fig. 8-9

Solutions:

(a) Using the Dunkerley equation.

$$\omega_1 = \sqrt{\frac{g}{\delta_{11}}} = \sqrt{\frac{g}{W_1 a_{11}}} = \sqrt{\frac{386}{(140)(2)10^{-6}}} = 1174 \text{ rad/sec}$$

$$\omega_2 = \sqrt{\frac{g}{\delta_{22}}} = \sqrt{\frac{g}{W_2 a_{22}}} = \sqrt{\frac{386}{(60)(12)10^{-6}}} = 732 \text{ rad/sec}$$

$$\frac{1}{\omega_c^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} = \frac{1}{(1174)^2} + \frac{1}{(732)^2} \quad \text{or} \quad \omega_c = 621 \text{ rad/sec}$$

(b) Using the Rayleigh-Ritz equation.

$$\omega_c = \sqrt{\frac{g \sum W\delta}{\sum W\delta^2}}$$

$$\delta_1 = W_1 a_{11} + W_2 a_{12} = (140)(2)10^{-6} + (60)(4)10^{-6} = (5.20)10^{-4} \text{ in.}$$
$$\delta_2 = W_2 a_{22} + W_1 a_{21} = (60)(12)10^{-6} + (140)(4)10^{-6} = (12.80)10^{-4} \text{ in.}$$

	$W\delta$		$W\delta^2$
(1)	$(140)(5.20)10^{-4} = (7.28)10^{-2}$	(1)	$(7.28)(10^{-2})(5.20)10^{-4} = (37.9)10^{-6}$
(2)	$(60)(12.80)10^{-4} = (7.68)10^{-2}$	(2)	$(7.68)(10^{-2})(12.80)10^{-4} = (98.3)10^{-6}$
	$\Sigma = (14.96)10^{-2}$ lb-in		$\Sigma = (136.2)10^{-6}$ lb-in^2

$$\omega_c = \sqrt{\frac{(386)(14.96)10^{-2}}{(136.2)10^{-6}}} = 651 \text{ rad/sec}$$

The two solutions differ, as is to be expected. The Dunkerley equation underestimates and the Rayleigh-Ritz equation overestimates. Hence the true value lies between 621 and 651 rad/sec.

(c) Using the frequency equation.

$$\frac{1}{\omega^4} - (a_{11}m_1 + a_{22}m_2)\frac{1}{\omega^2} + (a_{11}a_{22} - a_{12}a_{21})m_1 m_2 = 0$$

$$(a_{11}m_1 + a_{22}m_2) = (2)(10^{-6})\left(\frac{140}{386}\right) + (12)(10^{-6})\left(\frac{60}{386}\right) = (2.59)10^{-6}$$

$$(a_{11}a_{22} - a_{12}a_{21})m_1 m_2 = \left[(2)(12) - (4)(4)\right]\frac{(140)(60)10^{-12}}{(386)^2} = (0.451)10^{-12}$$

Hence $\dfrac{1}{\omega^4} - (2.59)10^{-6}\dfrac{1}{\omega^2} + (0.451)10^{-12} = 0$, for which the smaller positive root is $\omega_c = 624$ rad/sec.

This is the true value of the critical speed (within sliderule accuracy). Thus for this particular example the Dunkerley equation is a better approximation than the Rayleigh-Ritz equation.

5. The steel shaft in Fig. 8-10 has two gears weighing 50 lb and 100 lb respectively attached as shown. Neglecting the mass of the shaft, determine the first critical speed.

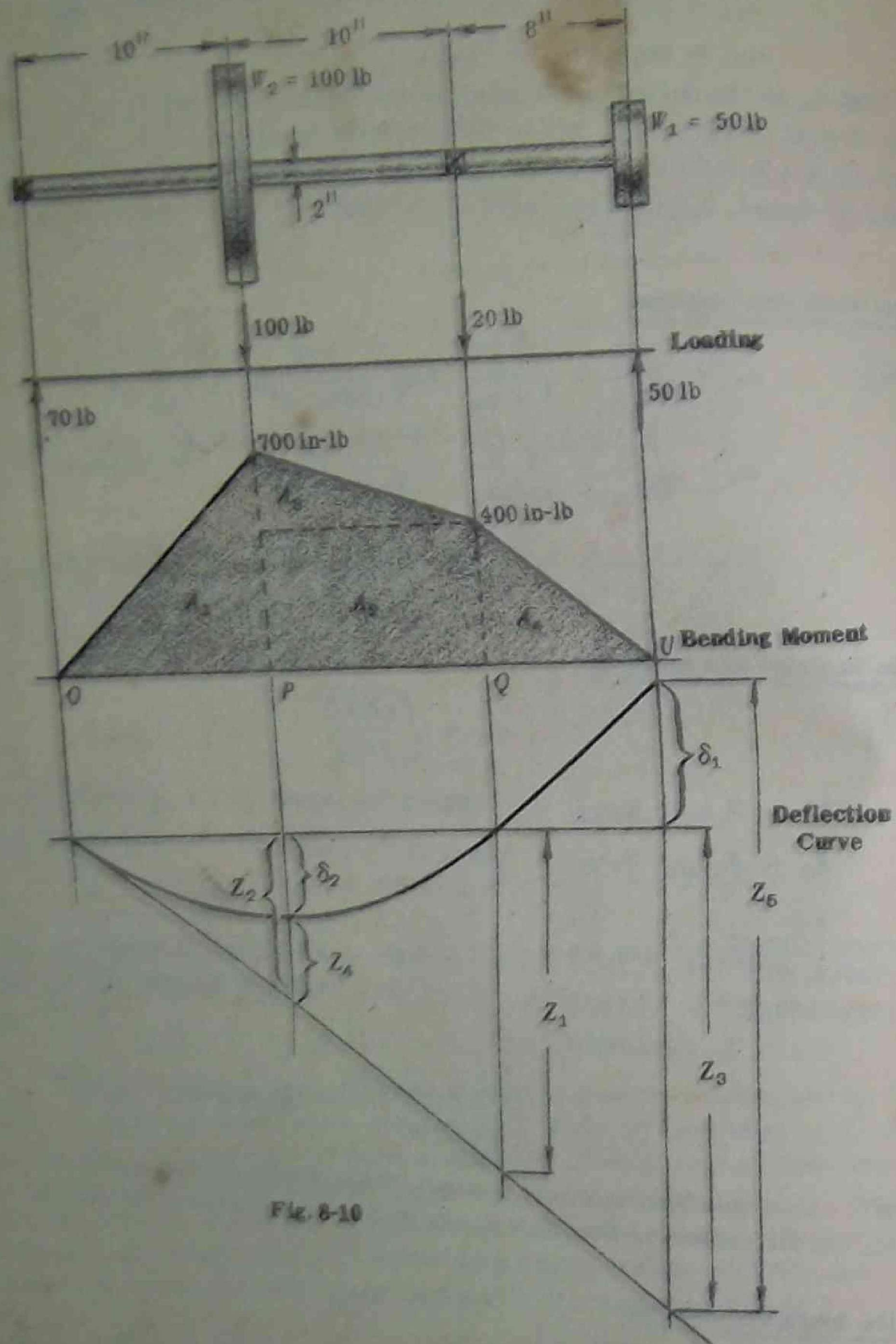


Fig. 8-10

Solution:

We choose to use the Rayleigh-Ritz equation, $\omega_c = \sqrt{\dfrac{g \sum W\delta}{\sum W\delta^2}}$. Most of the labor of computation lies in determining the δ's. We proceed as illustrated by the diagrams in Fig. 8-10 above.

(1) Assume static loading with forces equal in magnitude to W_1 and W_2 and so directed that the shaft will be bent to the simplest form possible (no reversal of curvature). This gives us the loading diagram shown.

(2) Calculate the bearing reactions, assuming point supports.

(3) Determine bending moments and sketch the bending moment diagram.

(4) Find deflections δ_1 and δ_2 by any suitable technique. Here we shall use the area-moment method. The necessary arithmetic is recorded below.

(a) Z_1EI = moment of areas A_1, A_2, A_3 around S

$$= \frac{(10)(700)}{2}(10 + \frac{10}{3}) + (10)(400)(5) + \frac{(20)(300)}{2}(\frac{20}{3}) = 76,667 \text{ lb-in}^3$$

(b) $Z_2EI = 76,667(10/20) = 38,333 \text{ lb-in}^3$

(c) $Z_3EI = 76,667(28/20) = 107,334 \text{ lb-in}^3$

(d) Z_4EI = moment of area A_1 around $P = \frac{(10)(700)}{2}(\frac{10}{3}) = 11,667 \text{ lb-in}^3$

(e) Z_5EI = moment of areas A_1, A_2, A_3, A_4 around U

$$= \frac{(10)(700)}{2}(18 + \frac{10}{3}) + (10)(400)(13) + \frac{(20)(300)}{2}(8 + \frac{20}{3}) + \frac{(8)(400)}{2}(\frac{16}{3})$$

$$= 157,216 \text{ lb-in}^3$$

(f) $\delta_2 EI = Z_2EI - Z_4EI = 38,333 - 11,665 = 26,666$

(g) $\delta_1 EI = Z_5EI - Z_3EI = 157,216 - 107,334 = 49,882$

(5) $I = \frac{\pi d^4}{64} = \frac{\pi(2)^4}{64} = 0.785 \text{ in}^4$, $\quad E = (3)10^7$ psi (steel)

(6) $\delta_1 = 49,882/(3)(10^7)(0.785) = (2.118)10^{-3}$ in.
$\quad \delta_2 = 26,666/(3)(10^7)(0.785) = (1.132)10^{-3}$ in.

(7) $W_1\delta_1 = (50)(2.118)10^{-3} = (10.59)10^{-2}$ $\qquad W_1\delta_1^2 = (2.243)10^{-4}$

$W_2\delta_2 = (100)(1.132)10^{-3} = (11.32)10^{-2}$ $\qquad W_2\delta_2^2 = (1.282)10^{-4}$

$\Sigma = (21.91)10^{-2}$ $\qquad \Sigma = (3.525)10^{-4}$

$$\omega_c = \sqrt{\frac{g\Sigma W\delta}{\Sigma W\delta^2}} = \sqrt{\frac{(386)(21.91)10^{-2}}{(3.525)10^{-4}}} = 490 \text{ rad/sec}$$

6. Determine both the first and second critical speeds for the system of Fig. 8-10.

Solution:

1. We shall use the frequency equation

$$\frac{1}{\omega^4} - (a_{11}m_1 + a_{22}m_2)\frac{1}{\omega^2} + (a_{11}a_{22} - a_{12}a_{21})m_1m_2 = 0$$

Most of the calculation labor is involved in the determination of the influence coefficients, a_{11}, a_{22}, $a_{12} = a_{21}$. Two deflection analyses must be made.

2. To determine a_{11} and a_{21} we apply a 1 lb load at the location of mass No. 1 and solve for deflections at the locations of masses No. 1 and No. 2 respectively (see Fig. 8-11). Similarly, to determine a_{22} and a_{12} we apply a 1 lb load at the location of mass No. 2 and solve for deflections at the locations of masses No. 2 and No. 1 respectively. The arithmetic will not be reproduced here. The results are

$$a_{12} = a_{21} = (8.50)10^{-6} \text{ in/lb},$$
$$a_{11} = (25.35)10^{-6} \text{ in/lb},$$
$$a_{22} = (7.07)10^{-6} \text{ in/lb}$$

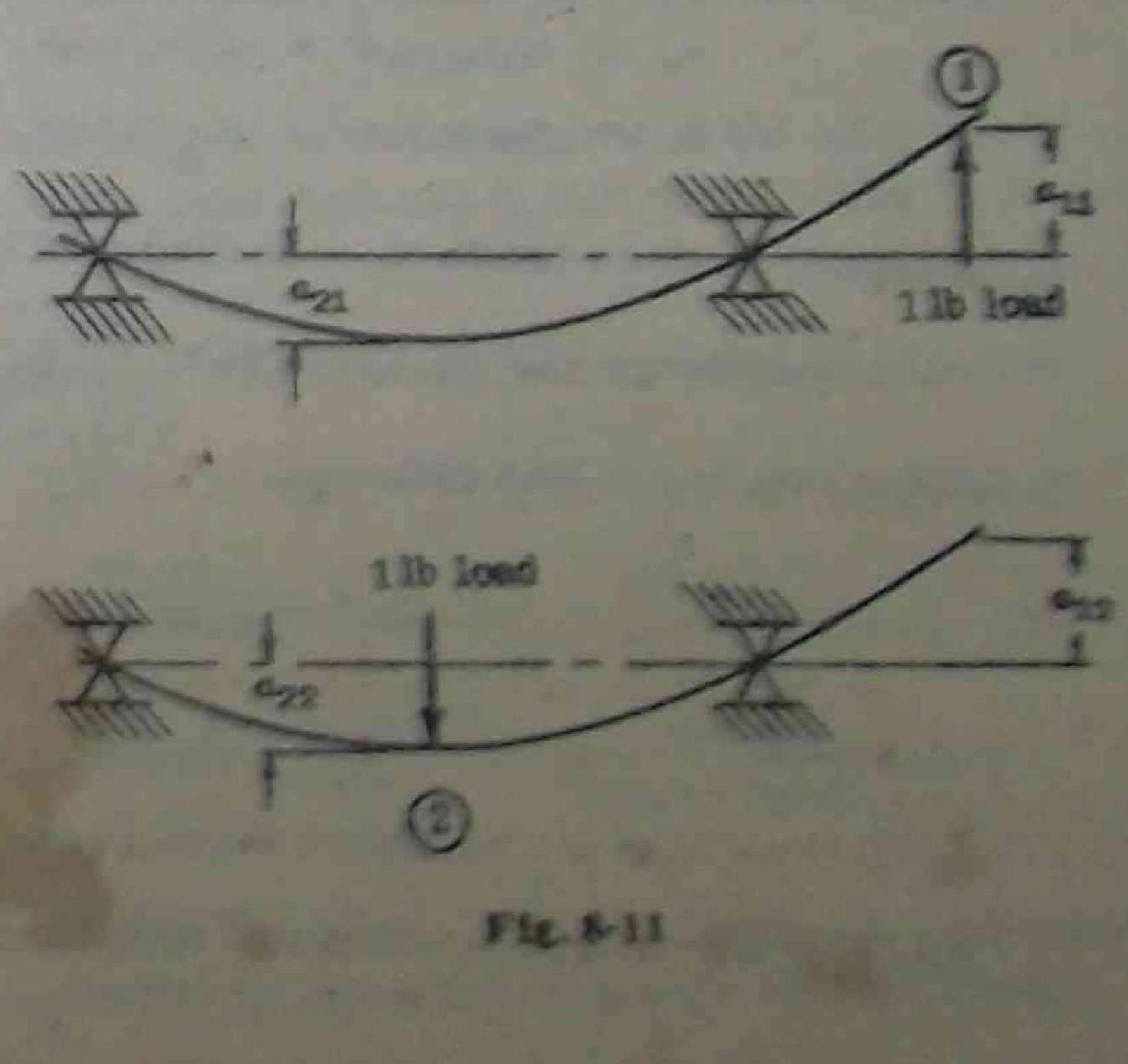


Fig. 8-11

3. $a_{11}m_1 + a_{22}m_2 = (25.35)10^{-6}(50/386) + (7.07)10^{-6}(100/386) = (5.115)10^{-6}$

$(a_{11}a_{22} - a_{12}a_{21})m_1m_2 = \left[(25.35)(7.07) - (8.50)^2\right](50)(100)(10^{-12})/(386)^2 = (3.59)10^{-12}$

4. Hence $\frac{1}{\omega^4} - (5.115)(10^{-6})\frac{1}{\omega^2} + (3.59)10^{-12} = 0$ having positive roots $\omega_{c_1} = 483$ and $\omega_{c_2} = 1090$ rad/sec

7. The bearing supports for the shaft shown in Fig. 8-12 have flexibility equivalent to a spring constant k of 250,000 lb/in in any direction perpendicular to the shaft axis. Due to bending, the shaft itself has a deflection δ_b of .0018 in. under the 300 lb load. What effect does the flexibility of the supports have on the critical speed?

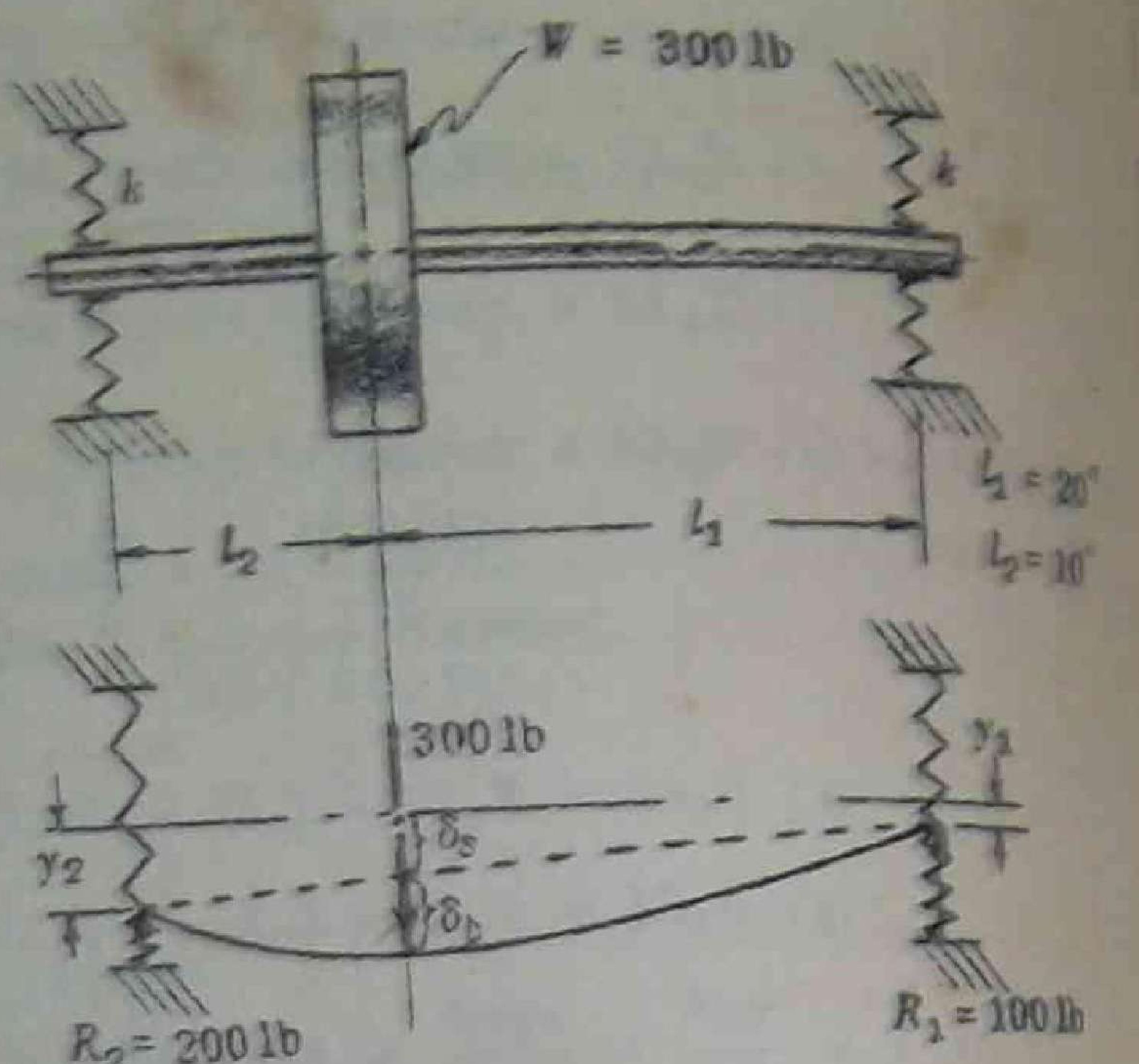


Fig. 8-12

Solution:

1. If the supports were completely rigid, the critical speed would be

$$\omega_c = \sqrt{g/\delta_b} = \sqrt{386/.0018} = 463 \text{ rad/sec}$$

2. The flexibility of the supports increases the deflection at the load, measured with respect to the unloaded shaft centerline. To calculate the critical speed we should use

$$\omega_c = \sqrt{g/(\delta_b + \delta_s)}$$

$y_1 = R_1/k = 100/250{,}000 = (4)10^{-4}$ in. $\qquad \delta_s = y_1 + (y_2 - y_1)\frac{l_1}{l_1 + l_2} = (6.7)10^{-4}$ in.

$y_2 = R_2/k = 200/250{,}000 = (8)10^{-4}$ in. $\qquad \delta_b + \delta_s = (18.0 + 6.7)10^{-4} = (24.7)10^{-4}$ in.

Then $\omega_c = \sqrt{386/(24.7 \times 10^{-4})} = 395$ rad/sec. The flexibility of the supports reduces the critical speed by

$$\left(\frac{463 - 395}{463}\right)100\% = 15\%$$

8. Derive the frequency equation

$$\frac{1}{\omega^4} - (a_{11}m_1 + a_{22}m_2)\frac{1}{\omega^2} + (a_{11}a_{22} - a_{12}a_{21})m_1m_2 = 0$$

for a two-mass system.

Solution:

1. Refer to Fig. 8-13. Consider the shaft rotating and deflected by the centrifugal forces $m_1y_1\omega^2$ and $m_2y_2\omega^2$ on the two masses.

$$y_1 = a_{11}m_1y_1\omega^2 + a_{12}m_2y_2\omega^2,$$
$$y_2 = a_{22}m_2y_2\omega^2 + a_{21}m_1y_1\omega^2$$

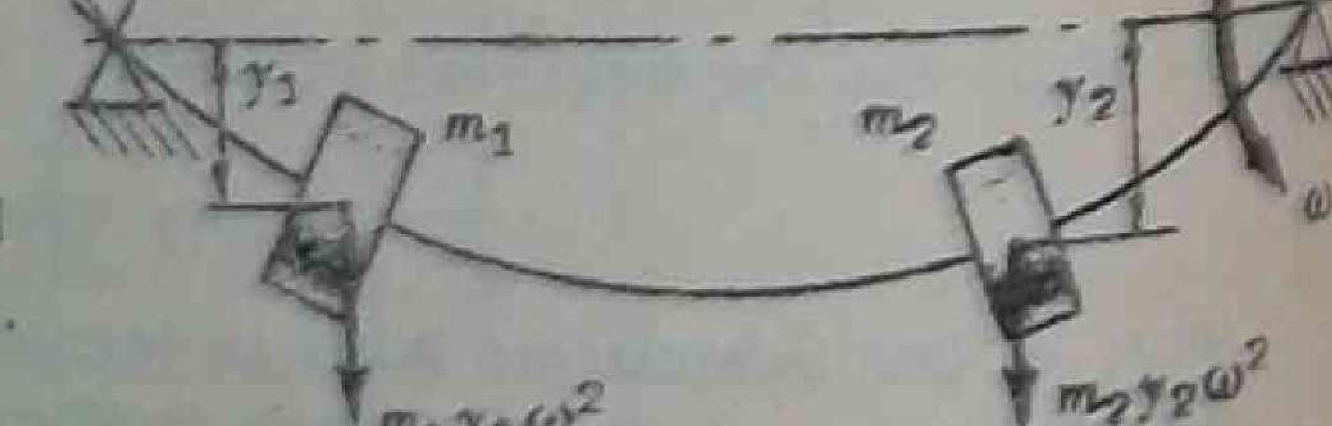


Fig. 8-13

2. Rearrange the above equations, collecting coefficients of y_1 and y_2, and dividing through by ω^2.

$$(a_{11}m_1 - 1/\omega^2)y_1 + (a_{12}m_2)y_2 = 0, \qquad (a_{21}m_1)y_1 + (a_{22}m_2 - 1/\omega^2)y_2 = 0$$

3. Solving for y_1/y_2 in both equations,

$$\frac{y_1}{y_2} = \frac{a_{12}m_2}{1/\omega^2 - a_{11}m_1}, \qquad \frac{y_1}{y_2} = \frac{1/\omega^2 - a_{22}m_2}{a_{21}m_1}$$

Then $\frac{a_{12}m_2}{1/\omega^2 - a_{11}m_1} = \frac{1/\omega^2 - a_{22}m_2}{a_{21}m_1}$, which can be rearranged into the form

$$\frac{1}{\omega^4} - (a_{11}m_1 + a_{22}m_2)\frac{1}{\omega^2} + (a_{11}a_{22} - a_{12}a_{21})m_1m_2 = 0$$

4. We might more simply have said that, in order that the two equations as written in (2) be satisfied, the determinant of the coefficients of y_1 and y_2 must vanish.

$$\begin{vmatrix} (a_{11}m_1 - 1/\omega^2) & (a_{12}m_2) \\ (a_{21}m_1) & (a_{22}m_2 - 1/\omega^2) \end{vmatrix} = 0$$

5. To develop the frequency equation with more masses, we could follow exactly the same procedure, writing one equation for the deflection at each mass. In order that the set of equations be satisfied, the determinant of the coefficients of the y's must vanish.

9. Derive the Dunkerley equation for a two-mass system.

Solution:

1. We start with the frequency equation derived in Problem 8:

$$\frac{1}{\omega^4} - (a_{11}m_1 + a_{22}m_2)\frac{1}{\omega^2} + (a_{11}a_{22} - a_{12}a_{21})m_1m_2 = 0$$

2. In any equation of the form $x^2 + bx + c = 0$, the sum of the roots is $-b$: $x_1 + x_2 = -b$. Hence in our frequency equation

$$(1/\omega_{c_1}^2 + 1/\omega_{c_2}^2) = a_{11}m_1 + a_{22}m_2$$

where ω_{c_1} and ω_{c_2} are the first and second critical speeds, respectively.

3. ω_{c_2} is larger than ω_{c_1}, usually appreciably larger. Then $1/\omega_{c_1}^2$ will generally be much larger than $1/\omega_{c_2}^2$. Therefore, *approximately*, $1/\omega_{c_1}^2 = a_{11}m_1 + a_{22}m_2$.

4. Now, $a_{11}m_1 = a_{11}W_1/g$; and $a_{11}W_1 = \delta_{11}$, the static deflection at mass No. 1 caused by W_1 acting alone. Hence $a_{11}m_1 = \delta_{11}/g = 1/\omega_1^2$, where ω_1 = critical speed that would exist if only mass No. 1 were present. Similarly, $a_{22}m_2 = 1/\omega_2^2$.

5. Thus $1/\omega_{c_1}^2 = 1/\omega_1^2 + 1/\omega_2^2$, approximately, the Dunkerley equation.

6. We can now see the reason for a previous statement to the effect that the Dunkerley equation underestimates the critical speed. The Dunkerley equation assumes $1/\omega_{c_1}^2 = a_{11}m_1 + a_{22}m_2$, whereas actually $1/\omega_{c_1}^2 = a_{11}m_1 + a_{22}m_2 - 1/\omega_{c_2}^2$.

10. A bare steel shaft of diameter D shows a first critical speed of 1200 rpm. If the shaft were bored to make it hollow, with an inside diameter of $\frac{3}{4}D$, what would be the critical speed?

Solution:

1. ω_c^2 is proportional to $1/\delta$; then $\omega_{ch}^2/\omega_{cs}^2 = \delta_s/\delta_h$, where ω_{ch} is the critical speed of the hollow shaft and ω_{cs} that of the solid shaft, δ_s the static deflection of the solid shaft and δ_h that of the hollow shaft (both measured at the same location).

2. Boring out the shaft reduces both the weight and stiffness, thus affecting the deflection in two ways.

The weight is reduced in the ratio $\dfrac{W_h}{W_s} = \dfrac{D^2 - (\frac{3}{4}D)^2}{D^2} = \dfrac{7}{16}$.

The moment of inertia I of the cross section is reduced in the ratio $\dfrac{I_h}{I_s} = \dfrac{D^4 - (\frac{3}{4}D)^4}{D^4} = \dfrac{175}{256}$.

3. Since δ is proportional to W/I, then $\delta_s/\delta_h = (16/7)(175/256) = 1.562$ and

$$\omega_{ch} = \omega_{cs}\sqrt{\delta_s/\delta_h} = 1200\sqrt{1.562} = 1500 \text{ rpm}$$

The reduction of mass tends to raise the critical speed, while the reduction of stiffness tends to lower it. The mass is reduced more than the stiffness; hence the net effect is to raise the critical speed.

Chapter 9

Power Transmission Shafting

SHAFT DESIGN consists primarily of the determination of the correct shaft diameter to ensure satisfactory strength and rigidity when the shaft is transmitting power under various operating and loading conditions. Shafts are usually circular in cross section, and may be either hollow or solid.

DESIGN OF SHAFTS of ductile materials, based on strength, is controlled by the maximum shear theory. The following presentation is based on shafts of ductile material and circular cross section. Shafts of brittle material would be designed on the basis of the maximum normal stress theory. Shafting is usually subjected to torsion, bending and axial loads. For torsional loads, the torsional stress τ_{xy} is

$$\tau_{xy} = M_t r/J = 16M_t/\pi d^3 \qquad \text{for solid shafts}$$
$$\tau_{xy} = 16M_t d_o/\pi(d_o^4 - d_i^4) \qquad \text{for hollow shafts}$$

For bending loads, the bending stress s_b (tension or compression) is

$$s_b = M_b r/I = 32M_b/\pi d^3 \qquad \text{for solid shafts}$$
$$s_b = 32M_b d_o/\pi(d_o^4 - d_i^4) \qquad \text{for hollow shafts}$$

For axial loads, the tensile or compressive stress s_a is

$$s_a = 4F_a/\pi d^2 \qquad \text{for solid shafts}$$
$$s_a = 4F_a/\pi(d_o^2 - d_i^2) \qquad \text{for hollow shafts}$$

The ASME Code equation for a hollow shaft combines torsion, bending, and axial loads by applying the maximum shear equation modified by introducing shock, fatigue, and column factors as follows:

$$d_o^3 = \frac{16}{\pi s_s(1-K^4)} \sqrt{\left[K_b M_b + \frac{\alpha F_a d_o(1+K^2)}{8}\right]^2 + (K_t M_t)^2}$$

For a solid shaft having little or no axial loading, the Code equation reduces to

$$d^3 = \frac{16}{\pi s_s} \sqrt{(K_b M_b)^2 + (K_t M_t)^2}$$

where, at the section under consideration,

τ_{xy} = torsional shear stress, psi

M_t = torsional moment, in-lb

M_b = bending moment, in-lb

d_o = shaft outside diameter, in.

d_i = shaft inside diameter, in.

F_a = axial load, lb

$K = d_i/d_o$

K_b = combined shock and fatigue factor applied to bending moment

K_t = combined shock and fatigue factor applied to torsional moment

	K_b	K_t
For stationary shafts:		
Load gradually applied	1.0	1.0
Load suddenly applied	1.5 to 2.0	1.5 to 2.0
For rotating shafts:		
Load gradually applied	1.5	1.0
Load suddenly applied (minor shock)	1.5 to 2.0	1.0 to 1.5
Load suddenly applied (heavy shock)	2.0 to 3.0	1.5 to 3.0

s_b = bending stress (tension or compression), psi

s_a = axial stress (tension or compression), psi

ASME Code specifies that for commercial steel shafting

s_s(allowable) = 8000 psi for shaft without keyway

s_s(allowable) = 6000 psi for shaft with keyway

ASME Code states that for steel purchased under definite specifications

s_s(allowable) = 30% of the elastic limit but not over 18% of the ultimate strength in tension for shafts without keyways. These values are to be reduced by 25% if keyways are present.

α = column-action factor. The column-action factor is unity for a tensile load. For a compression load, α may be computed by:

$$\alpha = \frac{1}{1 - 0.0044(L/k)} \qquad \text{for } L/k < 115$$

$$\alpha = \frac{s_y}{\pi^2 nE}\left(\frac{L}{k}\right)^2 \qquad \text{for } L/k > 115$$

n = 1 for hinged ends

n = 2.25 for fixed ends

n = 1.6 for ends partly restrained, as in bearings

k = radius of gyration = $\sqrt{I/A}$ in.

I = rectangular moment of inertia, in^4

A = cross section area of shaft, in^2

s_y = yield stress in compression, psi

DESIGN OF SHAFTS FOR TORSIONAL RIGIDITY is based on the permissible angle of twist. The amount of twist permissible depends on the particular application, and varies about 0.08 deg per foot for machine tool shafts to about 1.0 deg per foot for line shafting.

$$\theta = 584 M_t L / G(d_o^4 - d_i^4) \quad \text{for a hollow circular shaft}$$

$$\theta = 584 M_t L / G d^4 \quad \text{for a solid circular shaft}$$

where

θ = angle of twist, deg

L = length of shaft, in.

M_t = torsional moment, in-lb

G = torsional modulus of elasticity, psi

d = shaft diameter, in.

DESIGN OF SHAFTS FOR LATERAL RIGIDITY is based on the permissible lateral deflection for proper bearing operation, accurate machine tool performance, satisfactory gear tooth action, shaft alignment, and other similar requirements. The amount of deflection may be determined by two successive integrations of

$$d^2y/dx^2 = M_b/EI$$

where

M_b = bending moment, in-lb
E = modulus of elasticity, psi
I = rectangular moment of inertia, in⁴.

If the shaft is of variable cross section, a graphical solution of the above expression is practical. (See Chapter 5.)

STANDARD SIZES OF SHAFTING have been tentatively standardized by the American Engineering Standards Committee as follows:

Transmission shafting sizes in inches:

15/16, 1 3/16, 1 7/16, 1 11/16, 1 15/16, 2 3/16, 2 7/16, 2 15/16, 3 7/16, 3 15/16, 4 7/16, 4 15/16, 5 7/16, and 5 15/16.

Machinery shafting sizes in inches:

1/2 in. to 2 1/2 in. by 1/16 in. increments
2 5/8 in. to 4 in. by 1/8 in. increments
4 1/4 in. to 6 in. by 1/4 in. increments.

Standard stock lengths are 16, 20, and 24 ft.

BENDING AND TORSIONAL MOMENTS are the main factors influencing shaft design. One of the first steps in shaft design is to draw the bending moment diagram for the loaded shaft or the combined bending moment diagram if the loads acting on the shaft are in more than one axial plane. From the bending moment diagram, the points of critical bending stress can be determined.

The torsional moment acting on the shaft can be determined from

$$M_t = \frac{\text{hp} \times 33{,}000 \times 12}{2\pi\ \text{rpm}} = \frac{63{,}000 \times \text{hp}}{\text{rpm}} \quad \text{in-lb}$$

For a belt drive, the torque is found from

$$M_t = (T_1 - T_2)R \quad \text{in-lb}$$

where

T_1 = tight side of belt on pulley, lb
T_2 = loose side of belt on pulley, lb
R = radius of pulley, in.

For a gear drive, the torque is found from

$$M_t = F_t R$$

where

F_t = tangential force at the pitch radius, lb
R = pitch radius, in.

STRESSES DUE TO INTERFERENCE FITS may be calculated by considering the fitted parts as thick-walled cylinders, as shown in Fig. 3-2, by the following equations:

$$p_c = \frac{\delta}{d_c\left[\dfrac{d_c^2+d_i^2}{E_i(d_c^2-d_i^2)} + \dfrac{d_o^2+d_c^2}{E_o(d_o^2-d_c^2)} - \dfrac{\mu_i}{E_i} + \dfrac{\mu_o}{E_o}\right]}$$

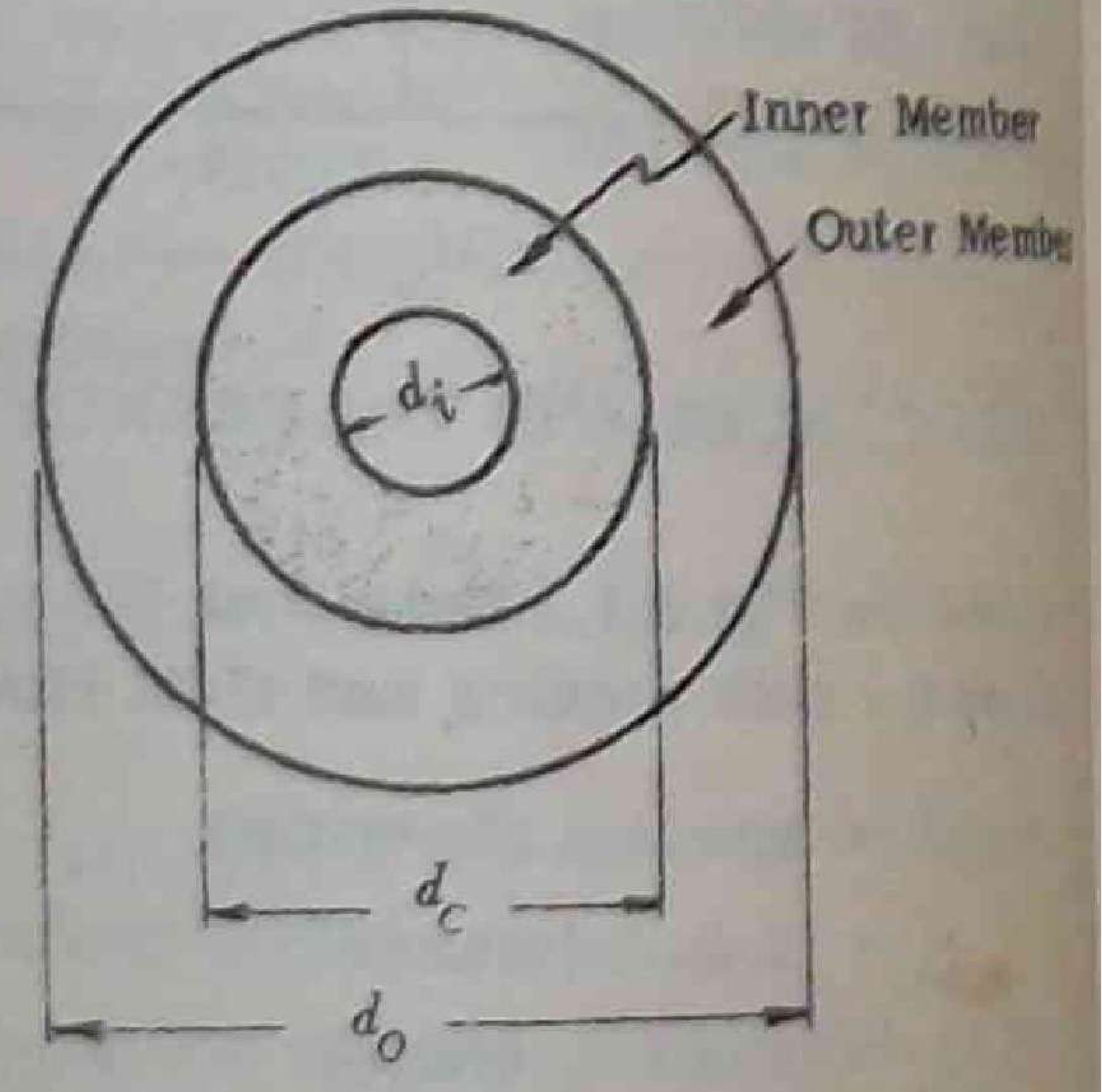


Fig. 3-2

where

p_c = pressure at the contact surface, psi
δ = the total interference, in.
d_i = inside diameter of the inner member, in.
d_c = diameter of the contact surface, in.
d_o = outside diameter of outer member, in.
μ_o = Poisson's ratio for outer member
μ_i = Poisson's ratio for inner member
E_o = modulus of elasticity of outer member, psi
E_i = modulus of elasticity of inner member, psi

If the outer and inner members are of the same material, the above equation reduces to

$$p_c = \frac{\delta}{\dfrac{2d_c^2(d_o^2-d_i^2)}{E(d_c^2-d_i^2)(d_o^2-d_c^2)}}$$

After p_c has been determined, then the actual tangential stresses at the various surfaces, in accordance with Lame's equation, for use in conjunction with the maximum shear theory of failure, may be determined by:

On the surface at d_o, $\qquad s_{to} = \dfrac{2p_c d_c^2}{d_o^2-d_c^2}$

On the surface at d_c for the outer member, $\qquad s_{tco} = p_c\left(\dfrac{d_o^2+d_c^2}{d_o^2-d_c^2}\right)$

On the surface at d_c for the inner member, $\qquad s_{tci} = -p_c\left(\dfrac{d_c^2+d_i^2}{d_c^2-d_i^2}\right)$

On the surface at d_i, $\qquad -2p_c d_c^2$

SOLVED PROBLEMS

1. A 3 ft length of commercial steel shafting is to transmit 50 hp at 3600 rpm through flexible cou- pling from an A.C. motor to a D.C. generator. Determine the required shaft size.

Solution:

In this case the shaft has only torsional stress and K_t is unity assuming that the load is gradually appli-

s_s(allowable) = 6000 psi in accordance with ASME Code for commercial shaft with keyway.

s_s(allowable) = $16M_t/\pi d^3$

$$6000 = 16\left(\frac{50 \times 63{,}000}{3600}\right)\left(\frac{1}{\pi d^3}\right)$$ or $d = 0.905$ in. Use 15/16 in., nearest standard size.

2. A section of commercial shafting 5 ft long between bearings carries a 200 lb pulley at its midpoi as shown in Fig. 9-1 below. The pulley is keyed to the shaft and receives 20 hp at 150 rpm wh is transmitted to a flexible coupling just outside the right bearing. The belt drive is horizontal the sum of the belt tensions is 1500 lb. Assume $K_t = K_b = 1.5$. Calculate the necessary shaft ameter and determine the angle of twist between bearings. $G = 12 \times 10^6$ psi.

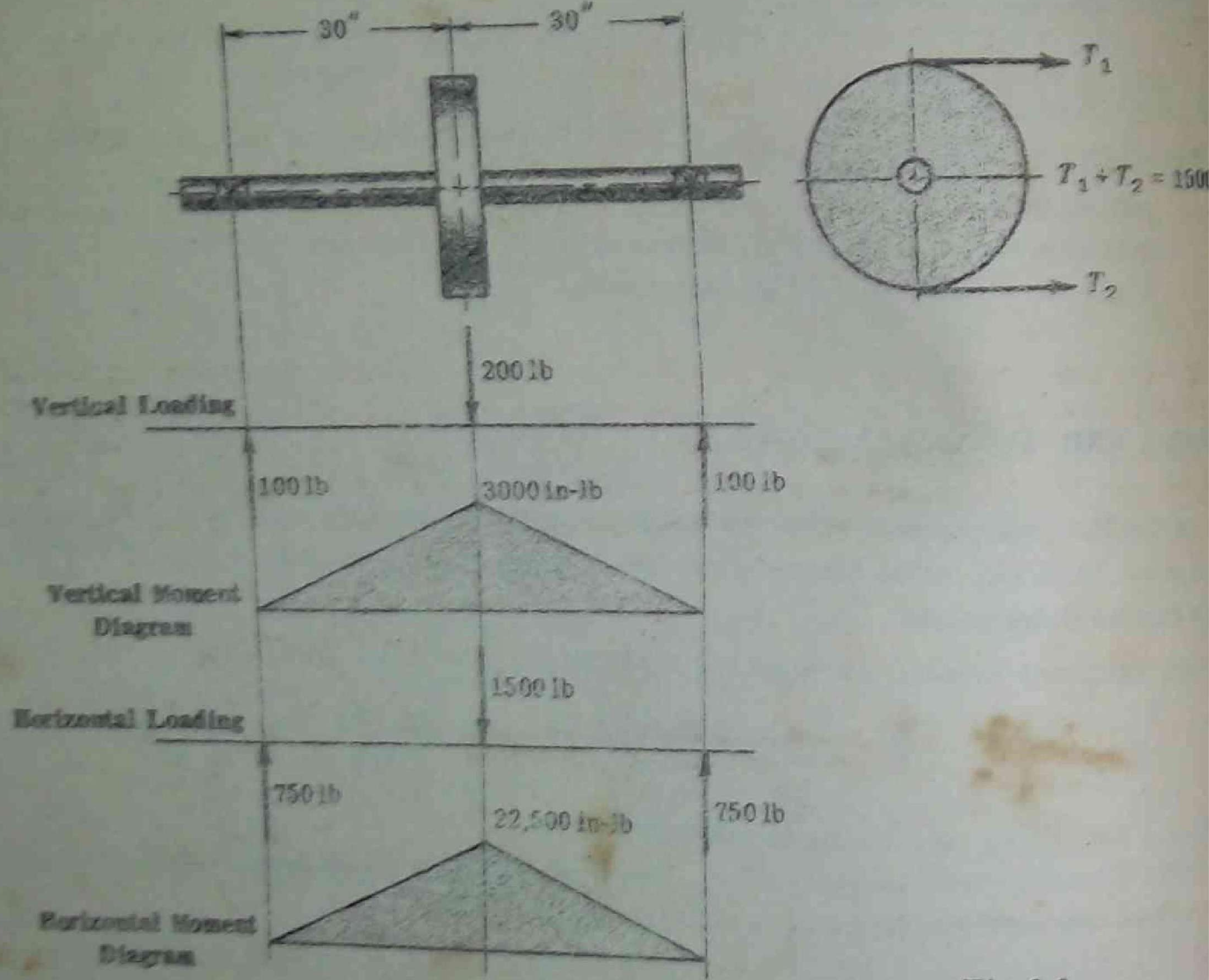


Fig. 9-1

Solution:

It is first necessary to determine the maximum bending and torsional moments acting on the shaft.

$M_b(\max) = \sqrt{3000^2 + 22{,}500^2} = 22{,}700$ in-lb $\qquad M_t(\max) = 20(63{,}000)/150 = 8400$ in-lb

s_s(allowable) = 6000 psi per ASME Code for shaft with keyway. Then

$$d^3 = \frac{16}{\pi s_s}\sqrt{(K_b M_b)^2 + (K_t M_t)^2} = \frac{16}{\pi 6000}\sqrt{(1.5 \times 22{,}700)^2 + (1.5 \times 8400)^2}$$

from which $d = 3.12$ in. Use 3 1/8 in. shaft, nearest standard size.

$$\theta = \frac{584 M_t L}{G d^4} = \frac{584 \times 8400 \times 30}{(12 \times 10^6)(3.125)^4} = 0.128^\circ \text{ twist}$$

3. Fig. 9-2 shows the forces acting on a steel shaft carrying two gears. The gears are keyed at B and D. A and C are journal bearing centers. Nine hp is transmitted at 650 rpm of the shaft. The allowable stress for an unkeyed section is 12,000 psi according to the ASME Code. $K_b = K_t = 1.5$.

(a) Sketch horizontal, vertical and resultant bending moment diagrams. Show values at change points.

(b) Determine the necessary shaft diameter to the nearest 0.01 in. Indicate the critical section.

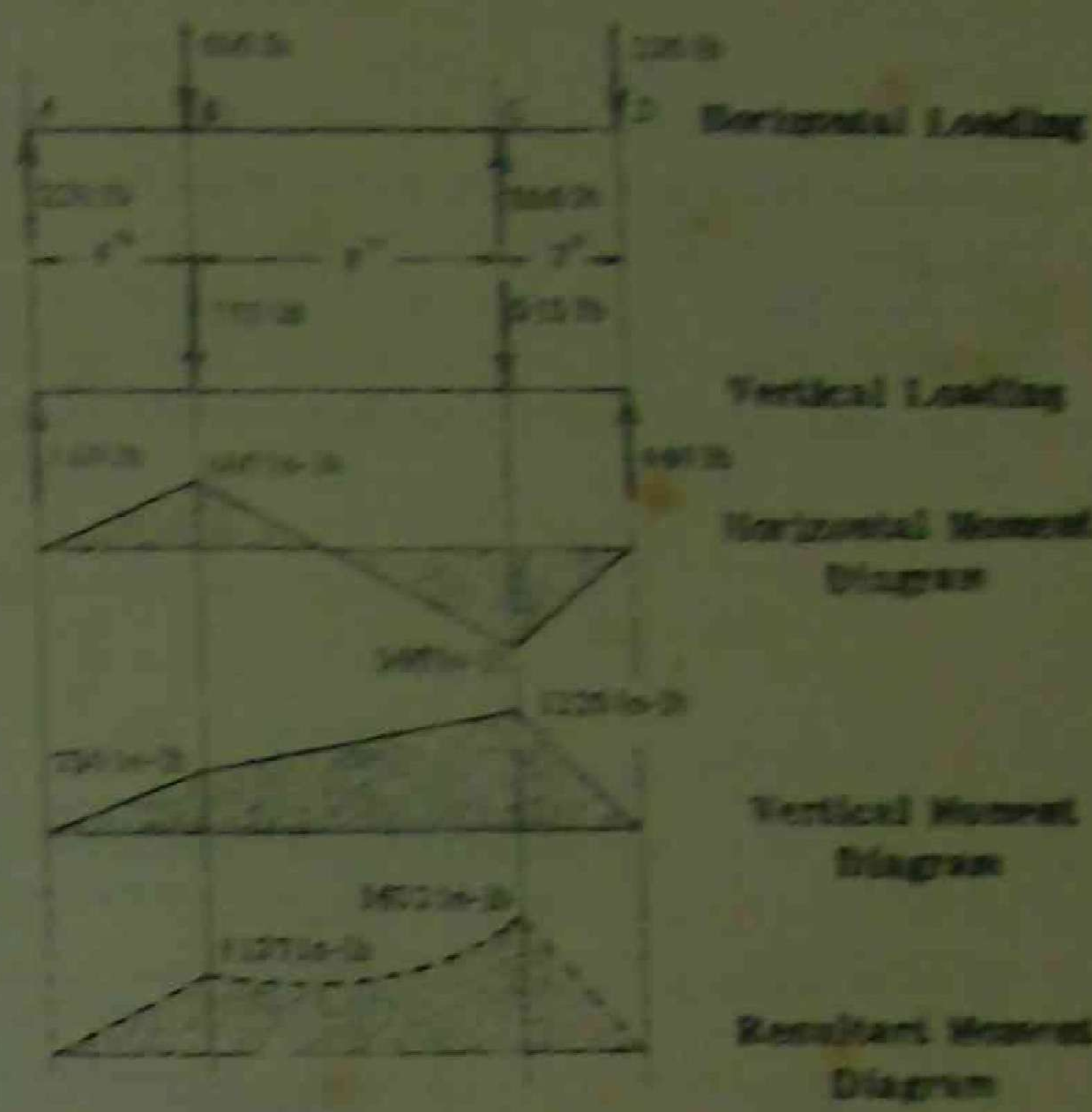


Fig. 9-2

Solution:

<u>At the bearing C:</u>

$$M_t(\text{max}) = 9(63,000)/650 = 872 \text{ in-lb}$$

$$M_b(\text{max}) = 1632 \text{ in-lb at } C$$

$$d^3 = \frac{16}{\pi(12,000)}\sqrt{(1.5\times 1632)^2 + (1.5\times 872)^2}$$

$$d = 1.07 \text{ in}$$

<u>Just to the right of gear B:</u>

$$M_t(\text{max}) = 872 \text{ in-lb}, \quad M_b(\text{max}) = 1132 \text{ in-lb}$$

$$d^3 = \frac{16}{\pi(0.75)(12,000)}\sqrt{(1.5\times 1132^2 + (1.5\times 872)^2} \quad \text{and} \quad d = 1.07 \text{ in}$$

Note: Even though the bending moment at gear B is less than at bearing C, the same size shaft is required as a result of the keyway at B.

4. A 24 inch diameter pulley driven by a horizontal belt transmits power through a solid shaft to a 10 inch diameter pinion which drives a mating gear. The pulley weighs 300 lb to provide some flywheel effect. The arrangement of elements, the belt tensions, and the components of the gear reactions on the pinion are as indicated in Fig. 9-3 below. Determine the necessary shaft diameter using ASME Code stress value for commercial shafting and shock fatigue factors of $K_b = 2$ and $K_t = 1.5$.

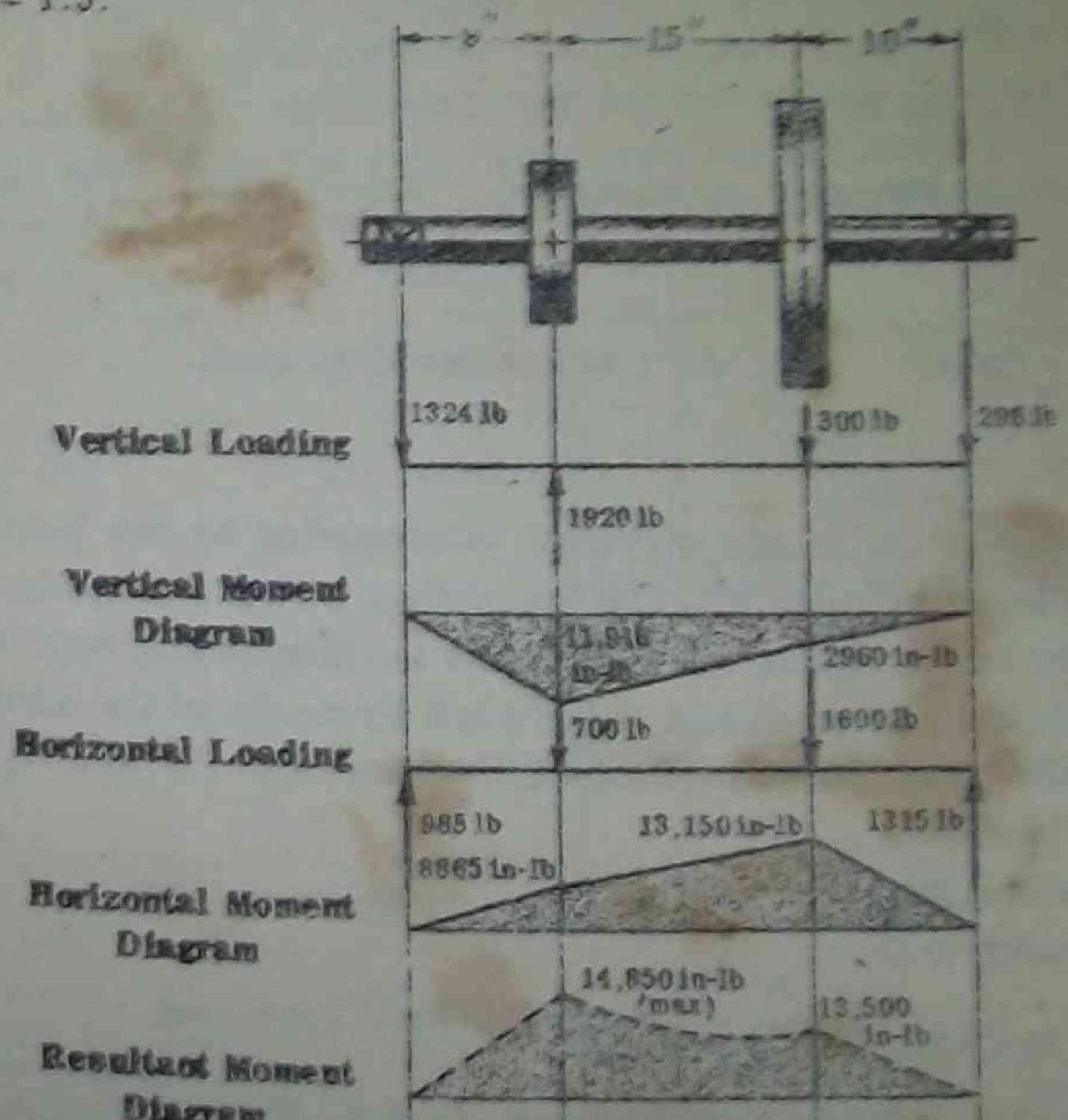


Fig. 9-3

Solution:

$$M_t(\text{max}) = (T_1 - T_2)12 = (1200 - 400)12 = 9600 \text{ in-lb}$$

$$M_b(\text{max}) = \sqrt{(11,916)^2 + (8865)^2} = 14,850 \text{ in-lb}$$

$$s_s(\text{allowable}) = 6000 \text{ psi per ASME Code}$$

$$d^3 = \frac{16}{\pi 6000}\sqrt{(2\times 14,850)^2 + (1.5\times 9600)^2} = 28$$

$d = 3.04$ in Use 3 inch diameter shaft.

5. A machine shaft turning at 600 rpm is supported on bearings 30 inches apart as shown in Fig. 9-4 below. Twenty hp is supplied to the shaft through an 18 inch pulley located 10 inches to the right of the right bearing. The power is transmitted from the shaft through an 8 inch spur gear located 10 inches to the right of the left bearing. The belt drive is at an angle of 60° above the horizontal. The pulley weighs 200 lb to provide some flywheel effect. The ratio of the belt tensions is 3:1. The gear has a 20° tooth form and mates with another gear located directly above the shaft. If the shaft material selected has an ultimate strength of 70,000 psi and a yield point of 46,000 psi, determine the necessary diameter in accordance with the ASME Code using $K_b = 1.5$ and $K_t = 1.0$.

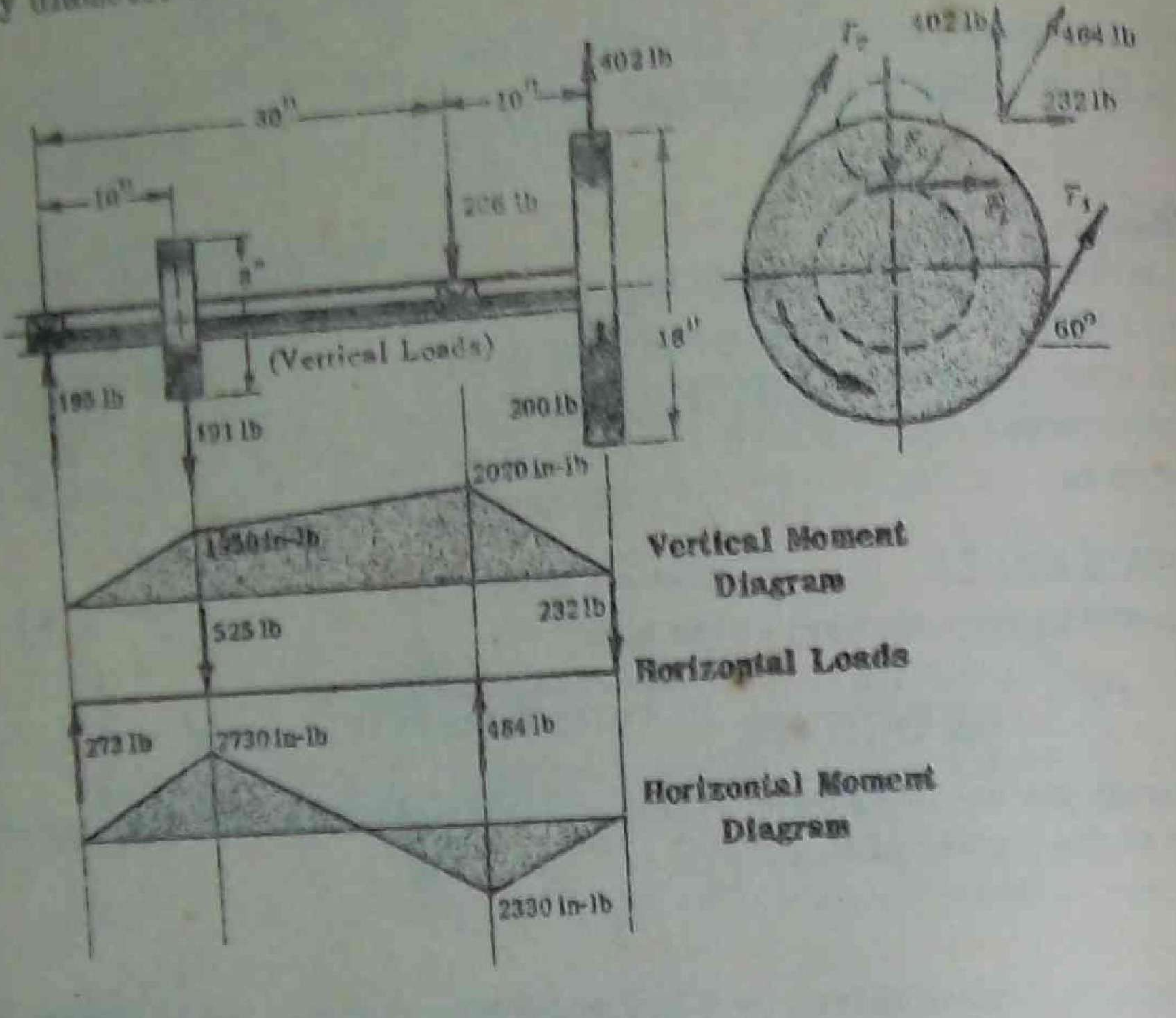


Fig. 9-4

Solution:

$M_t = 20(63{,}000)/600 = 2100$ in-lb

From $(T_1 - T_2)9 = 2100$ and $3T_1 = T_2$ we have

$T_2 = 116$ lb. $T_1 = 348$ lb. $(T_1 + T_2) = 464$ lb.

$4F_t = 2100$. $F_t = 525$ lb. $F_r = 525 \tan 20° = 191$ lb.

$18\% \times 70{,}000 = 12{,}600$ psi. $30\% \times 46{,}000 = 13{,}800$ psi. s_s(allowable) $= 75\% \times 12{,}600 = 9450$ psi.

$M_b(max) = \sqrt{1950^2 + 2730^2} = 3360$ in-lb

$$d^3 = \frac{16}{\pi 9450}\sqrt{(3360 \times 1.5)^2 + (2100)^2} \quad \text{or} \quad d = 1.44 \text{ in. Use } 1\tfrac{1}{2} \text{ in. shaft.}$$

6. A hollow shaft, 20 in. outside diameter and 12 in. inside diameter, is supported by two bearings 20 ft apart. The shaft is driven by a flexible coupling at one end and drives a ship's propeller at 100 rpm. The maximum thrust on the propeller is 120,000 lb when the shaft is transmitting 8000 hp. The shaft weighs 15,000 lb. Determine the maximum shear stress in the shaft by means of the ASME Code equation considering the weight of the shaft and the column effect. Assume $K_b = 1.5$ and $K_t = 1.0$.

Solution:

$M_b(max) = PL/8 = (15{,}000)(240)/8 = 450{,}000$ in-lb

$M_t(max) = (8000 \times 63{,}000)/100 = 5{,}040{,}000$ in-lb.

$I = \pi(20^4 - 12^4)/64 = 6840$ in^4. $A = \pi(20^2 - 12^2)/4 = 201$ in^2

$k = \sqrt{I/A} = \sqrt{6840/201} = 5.84$ in. $L/k = 240/5.84 = 41.2$, which is < 115. Then

$$\gamma = \frac{1}{1 - 0.0044(240/5.84)} = 1.222$$

$d_o = 20$ in., $d_i = 12$ in., and $K = d_i/d_o = 12/20 = 0.6$.

$$s_s = \frac{16}{\pi d_o^3(1-K^4)} \sqrt{\left[K_b M_b + \frac{\alpha F_a d_o (1+K^2)}{8}\right]^2 + (K_t M_t)^2}$$

$$= \frac{16}{\pi 20^3(1-0.6^4)} \sqrt{\left[(1.5 \times 450{,}000) + \frac{1.222 \times 120{,}000 \times 20(1+0.6^2)}{8}\right]^2 + (1 \times 5{,}040{,}000)^2} = 2800 \text{ psi}$$

7. A shaft 48 in. long receives 10,000 in-lb torque from a pulley located at the center of the shaft, as shown in Fig. 9-5. A gear at the left end of the shaft transmits 6000 in-lb of this torque from the shaft while the remainder is transmitted through a gear located at the right end of the shaft. Calculate the angular deflection of the left end of the shaft with respect to the right end of the shaft if the shaft is 2 in. in diameter and is made of steel. Neglect the effect of the keyways in the calculation.

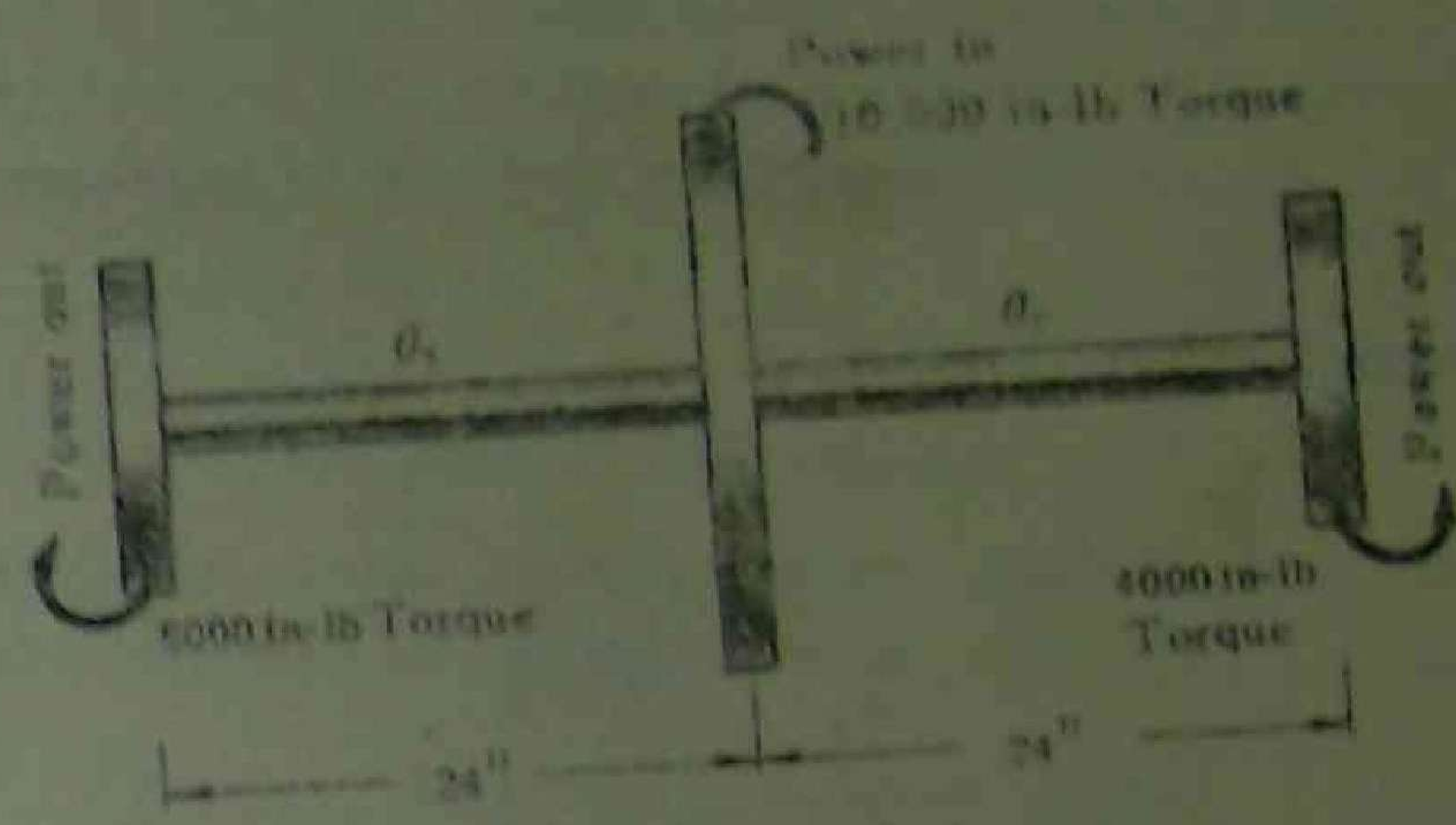


Fig. 9-5

Solution:

The angular deflection of one end of the shaft with respect to the other is the difference of the angular deflection of the ends with respect to the center.

$$\theta_1 - \theta_2 = \frac{584(6000)(24)}{Gd^4} - \frac{584(4000)(24)}{Gd^4} = \frac{584(24)(6000-4000)}{(12 \times 10^6)(2^4)} = 0.146^\circ$$

8. Thirty horsepower is supplied to the 30 in. sprocket by means of a chain drive as shown in Fig. 9-6. Twenty horsepower is taken off at the 24 in. pulley which weighs 1000 lb, and 10 hp is taken off at the 8 in. crank. The force in the chain on the tight side is represented by T_c. The tension in the slack side is so small that it can be neglected. The ratio of the tensions in the belt is 4:1. The shaft is rotating at 300 rpm. The loads are applied with moderate shock. $K_b = 2$ and $K_t = 1.5$. Determine the size of shaft necessary if s_s(allowable) = 8000 psi. It is assumed that the sprocket and pulley are keyed to the shaft.

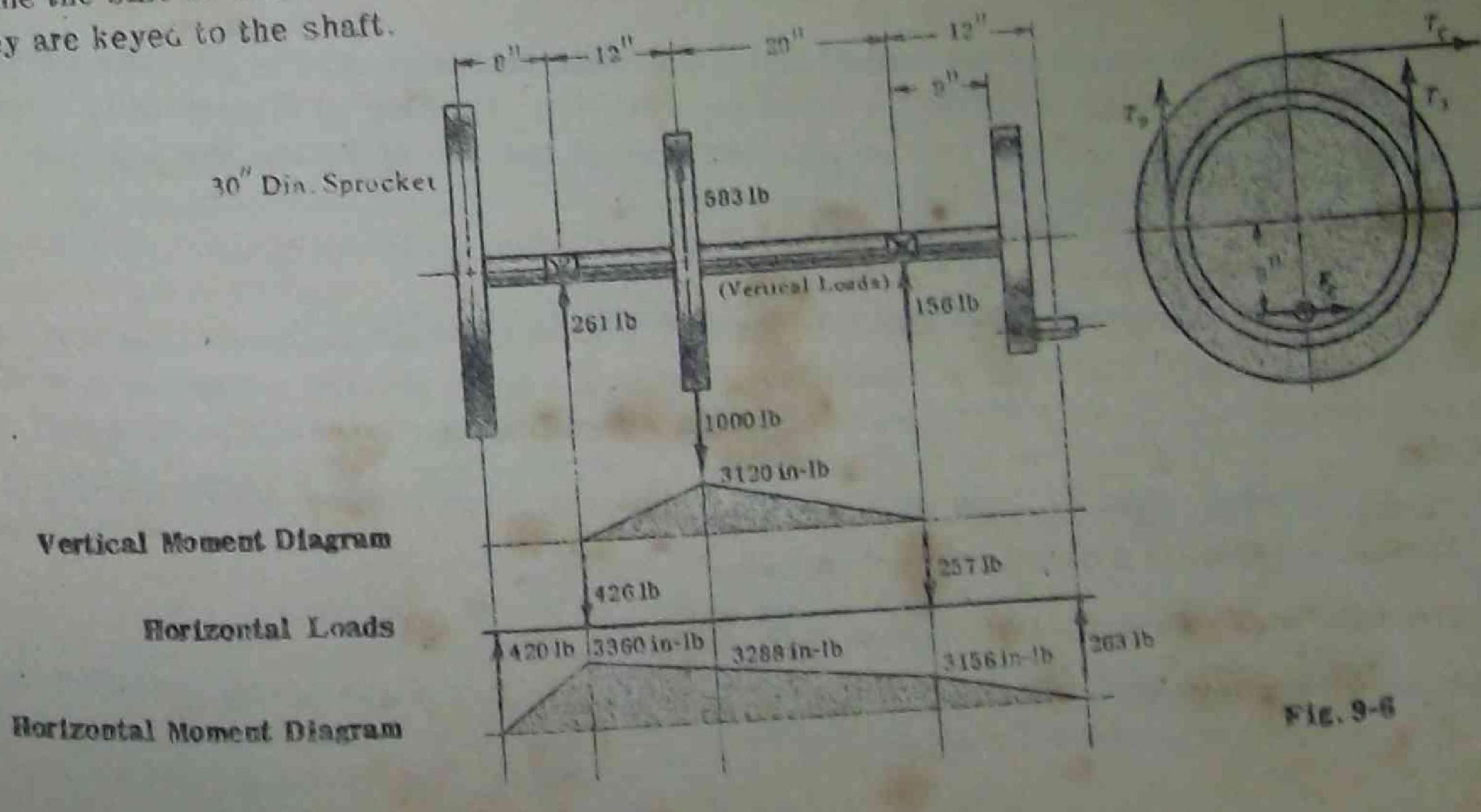


Fig. 9-6

From $(T_1 - T_2)12 = 4200$ and $T_1 = 4T_2$ we have $T_1 = 466$ lb. $T_2 = 117$ lb, and $(T_1 + T_2) = 583$ lb.

$F_c = 2100/8 = 263$ lb.

M_t(sprocket) $= 63{,}000 \times 30/300 = 6300$ in-lb. M_t(pulley) $= 2M_t$(crank) $= 4200$ in-lb. M_t(crank) $= 2100$ in-lb.

$M_b(\text{max}) = \sqrt{3288^2 + 3120^2} = 4490$ in-lb at pulley. $M_t(\text{max}) = 6300$ in-lb at pulley.

$$d^3 = \frac{16}{\pi s_s(0.75)}\sqrt{(K_b M_b)^2 + (K_t M_t)^2} = \frac{16}{\pi(8000)(0.75)}\sqrt{(2 \times 4490)^2 + (1.5 \times 6300)^2} = 11.06$$

from which $d = 2.23$ in. Use $2\frac{3}{16}$ in. shaft.

9. Determine the diameter below which the angle of twist of a shaft, and not the maximum stress, is the controlling factor in design of a solid shaft in torsion. The allowable shear stress is 8000 psi and the maximum allowable twist is 1/12 degree per foot. (Consider a shaft with no key). $G = 12 \times 10^6$ psi.

Solution:

$$s_s(\text{allowable}) = 16M_t/\pi d^3, \qquad \theta(\text{allowable}) = 584\,M_t L/Gd^4.$$

$M_t = \theta d^4 G/584L$ moment that can be transmitted in allowable twist.

$M_t = s_s \pi d^3/16$ moment that can be transmitted in allowable stress.

Then $\dfrac{\theta d^4 G}{584L} = \dfrac{s_s \pi d^3}{16}$ or $\dfrac{(1/12)(d^4)(12 \times 10^6)}{584(12)} = \dfrac{8000\,\pi d^3}{16}$, from which $d = 11.0$ in.

10. Shafts AB and CD are connected by spur gears as shown in Fig. 9-7. A couple applied at A stresses the shaft AB to 8000 psi. Determine the diameter of the shaft CD if the shearing stress therein must not exceed 8000 psi. Neglect keyways and any bending action. $G = 12 \times 10^6$ psi.

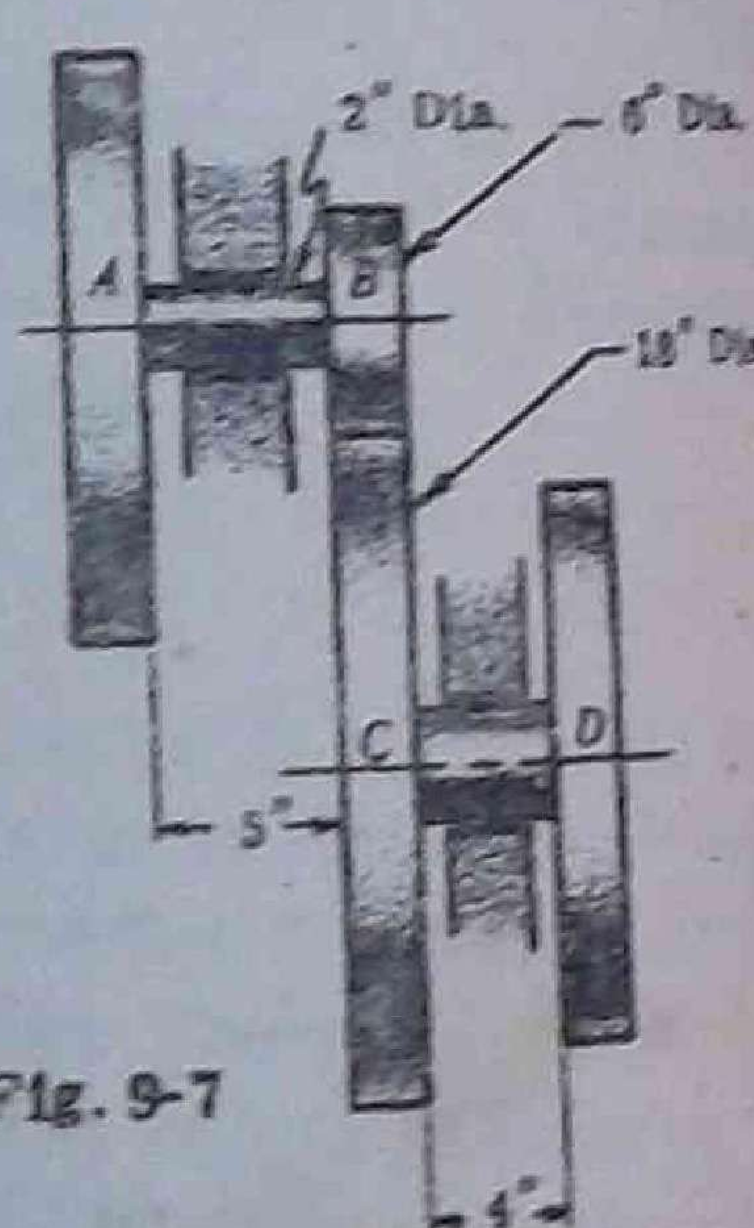


Fig. 9-7

Solution:

M_t on shaft CD is 3 times M_t on shaft AB.

Let diameter of shaft $AB = d_1$, of shaft $CD = d_2$. Then

$$s_s \pi d_1^3/16 = s_s \pi d_2^3/(16 \times 3)$$

and, since $d_1 = 2$ in., the required diameter $d_2 = 2.88$ in.

11. In order to reduce the weight of the control and power plant facilities used in a certain aircraft, it is planned to use all hollow shafting for power transmission. Develop an expression to determine the percent weight savings that may be effected through the use of hollow shafting in place of equal strength solid shafting for such an application.

Solution:

For a solid shaft subjected to torsional and bending loads, the shear stress is

$$s_s = \frac{16}{\pi d^3}\sqrt{M_b^2 + M_t^2}$$

And for a hollow shaft subjected to the same torsional and bending loads as above, the shear stress is

$$s_s = \frac{16\,d_o}{\pi(d_o^4 - d_i^4)}\sqrt{M_b^2 + M_t^2}$$

where M_b = bending moment at the critical section, in-lb; d_o = outside diameter of the hollow shaft, in

M_t = torsional moment at the critical section, in-lb; d_i = inside diameter of the hollow shaft, in

Since it is assumed that the hollow shafting will be of equal strength to that of the solid shafting, we may equate the right sides of the two above equations and obtain

$$\frac{16}{\pi d^3} = \frac{16\, d_o}{\pi(d_o^4 - d_i^4)} \qquad \text{from which} \qquad (1) \quad 1 - \left(\frac{d}{d_o}\right)^3 = \left(\frac{d_i}{d_o}\right)^4$$

From a weight consideration, the hollow shafting will be lighter than the solid shafting by a factor of $(1 - N/100)$, where N is the percent weight savings to be effected through the use of hollow vs solid shafting. Then

$$\frac{\pi}{4}(d_o^2 - d_i^2)L\lambda = \left(1 - \frac{N}{100}\right)\frac{\pi}{4}d^2 L\lambda \qquad \text{or} \qquad (2) \quad (d_o^2 - d_i^2) = \left(1 - \frac{N}{100}\right)d^2$$

where L = length of shafting, in.
λ = specific weight of shafting material, lb/in^3

Substituting the value of d_i from (1) into (2) and solving for N, we obtain

$$N = \left[1 - (d_o/d)^2 + \sqrt{(d_o/d)^4 - d_o/d}\,\right]100$$

12. The shaft shown in Fig. 9-8(a) is to be designed from the standpoint of strength, critical speed, and rigidity. Power is supplied to the pulley P by means of a flat belt and power is taken from the shaft through spur gear G. The shaft is supported by two deep groove ball bearings.

The following information has been established:

Horsepower = 10 (steady load conditions)
Speed of shaft = 900 rpm
Shaft is to be machined from hot rolled AISI 1035 (s_u = 85,000 psi and s_y = 55,000 psi)
Diameter of pulley = 10 in
Pitch diameter of the gear = 10 in.
Weight of the pulley = 30 lb
Weight of the gear = 30 lb
Ratio of belt tensions $T_1/T_2 = 2.5$
Gear pressure angle = 20°
The pulley and gear are assembled with press fits and keys.
Dimension $A = B = C = 6$ in.

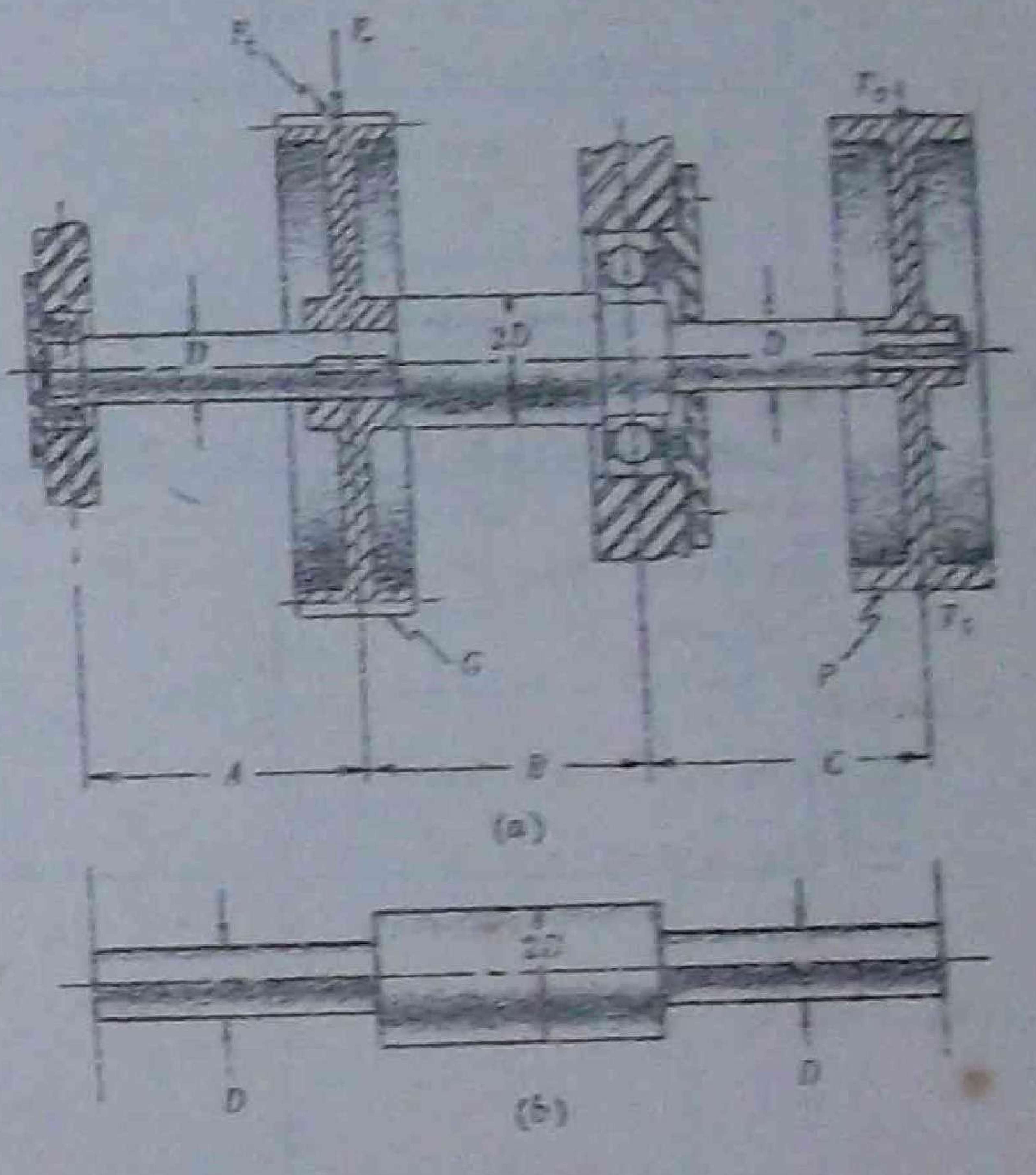


Fig. 9-8

The belt forces are perpendicular to the paper with the tight side being T_1 and the slack side T_2. The tangential force on the gear is F_t and the separating force is F_r. F_t is perpendicular to the paper.

The following limitations have been imposed:

(a) The shaft at the gear shall not deflect more than 0.001 in.
(b) The slope of the shaft through the bearings shall not exceed 1°.
(c) The operating speed of the shaft shall not be more than 60% of the lowest critical speed.

Solution:

The hub of the gear and of the pulley contribute to the stiffness of the shaft, as well as the inner race of each bearing. If a hub is relatively long, its effect is different than if the hub is short. For purposes of this problem, half lengths of hubs and inner races of the bearings will be considered effective in stiffening the shaft. The effect of the web of the pulley and gear can be neglected in deflection analysis. Design will then be based upon the simplified shaft as shown in Fig. 9-8(b). We will first determine the required diameter D for strength in accordance with the ASME shafting code.

The torque between the pulley and gear is M_t = (10)(63,000)/900 = 700 in-lb.

The sum of the belt tensions may be determined from

$$(T_1 - T_2)(5) = 700 \quad \text{and} \quad T_1 = 2.5\,T_2$$

from which T_1 = 233.3 lb, T_2 = 93.3 lb, and $(T_1 + T_2)$ = 326.6 lb.

The transmitted force F_t = 700/5 = 140 lb

The separating force F_r = 140 tan 20° = 51 lb.

From the above information the moment diagrams for vertical and horizontal loadings may be combined to obtain the resultant moment diagram as shown in Fig. 9-9 below.

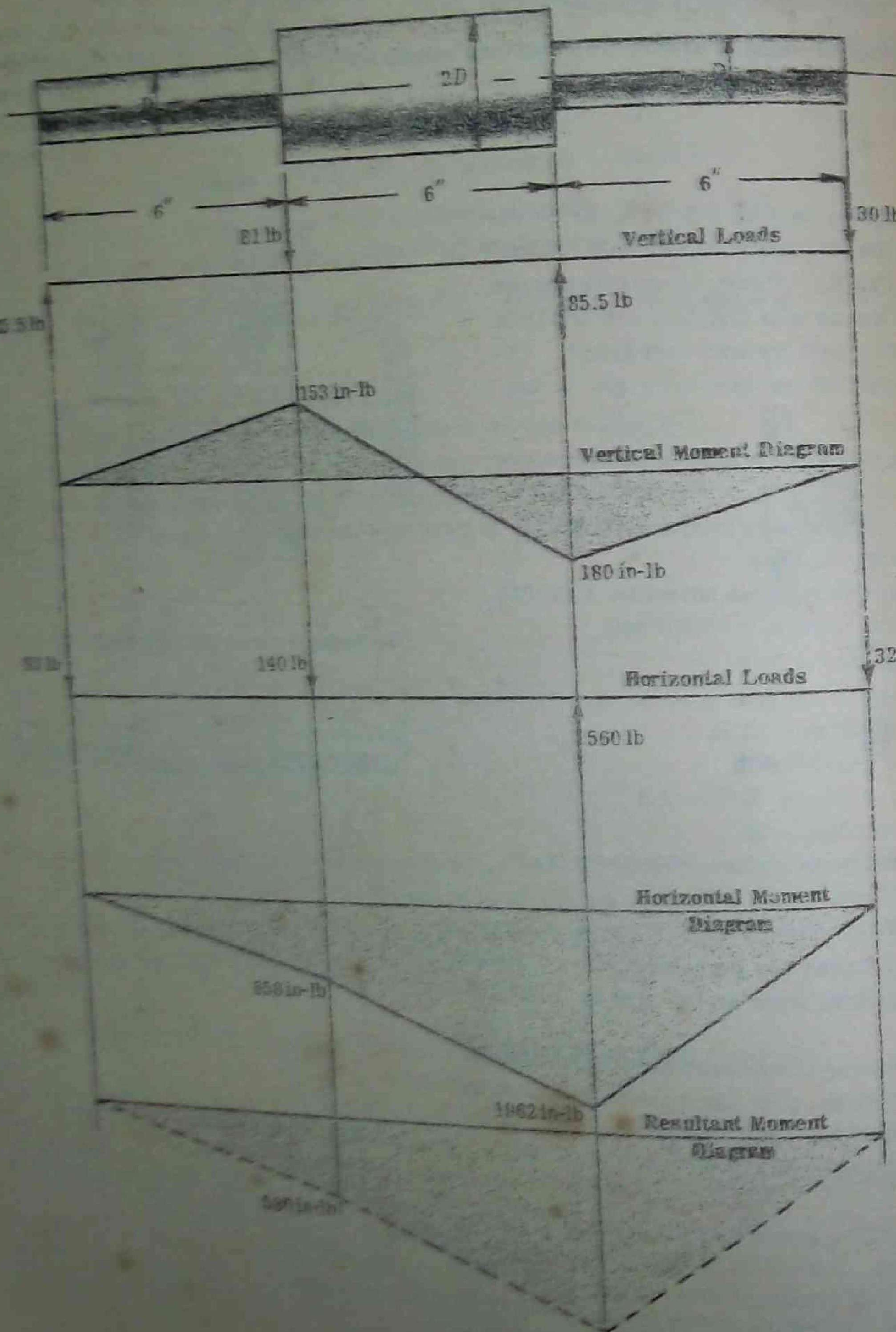


Fig. 9-9

The allowable shear stress is determined by: 18% × 85,000 = 15,300 psi; 30% × 55,000 = 16,500 psi. Use s(allowable) = 15,300 psi.

As seen from Fig. 9-9, the maximum resultant bending occurs at the right bearing: M_b = 1970 in-lb, M_t = 700 in-lb.

For steady load, $K_b = 1.5$ and $K_t = 1.0$.

$$D^3 = \frac{16}{\pi(15{,}300)}\sqrt{(1.5\times 1970)^2 + (1\times 700)^2} = 1.01$$

from which D = 1.00 in., and $2D$ = 2.00 in. required for strength.

Next, in order to determine the required diameter for operating below 60% of the first critical speed, it is necessary to compute the static deflections at the gear and pulley locations due only to the weights of the gear and pulley. It is important to note that in order to obtain the first critical speed, the weight of the pulley at the right end of the shaft has to be considered acting upward in order to satisfy the first mode of vibration. The shaft with its elastic curve and moment diagram for this purpose is shown in Fig. 9-10 below. Note that it is expedient to draw the tangent to the elastic curve at the left end in order to employ the area moment method for determining the required deflections. Instead of drawing the M/EI diagram for the following deflections, we will simply note that the moment of inertia for the $2D$ section is 16 times the moment of inertia of the D sections.

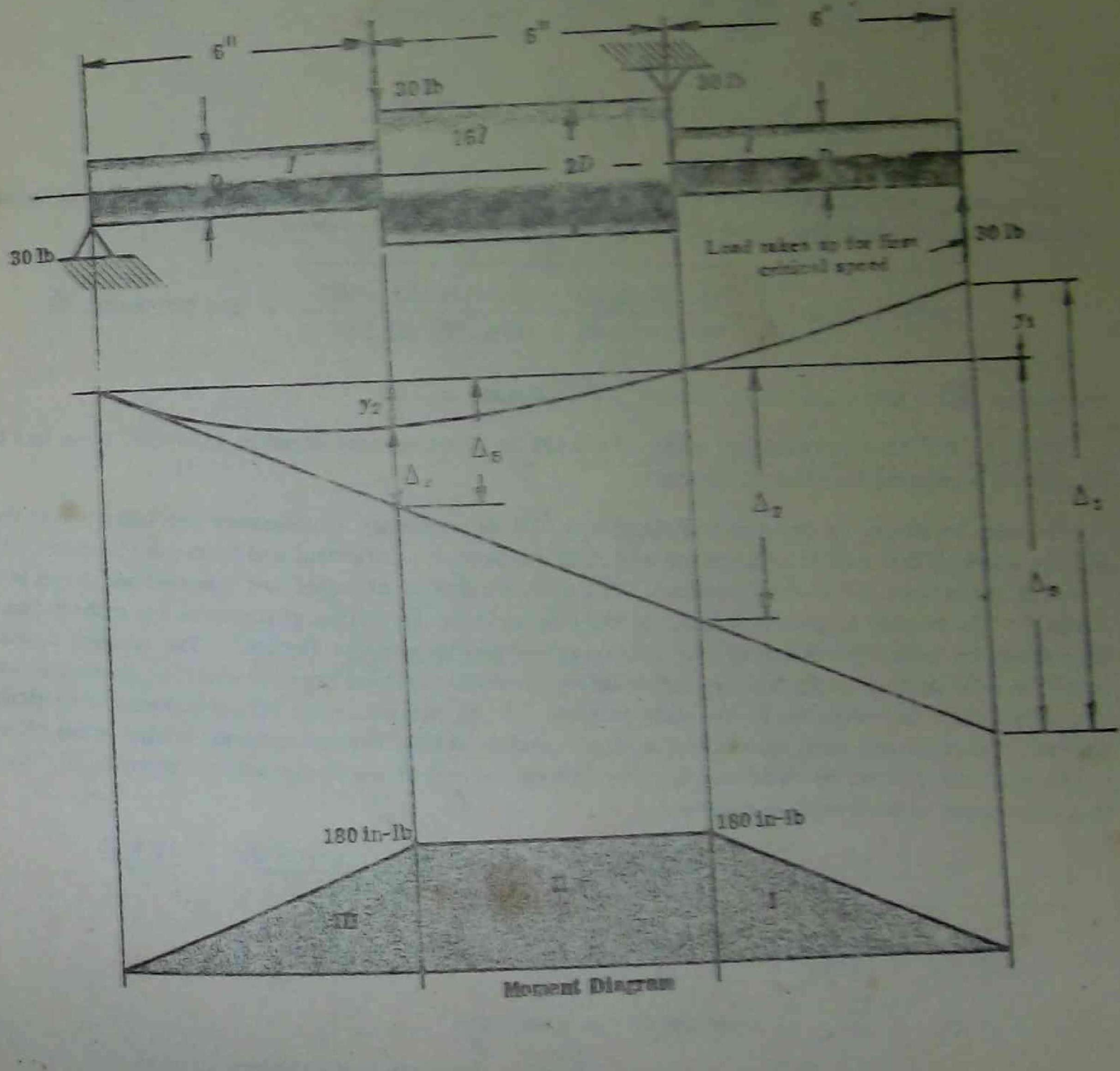


Fig. 9-10

Chapter 10

Coupling Design

COUPLINGS are used to connect sections of shafts or to connect the shaft of a driving machine to the shaft of a driven machine. This affords a permanent connection, contrasted with clutches which provide for engagement or disengagement at will. Clutches are treated in a different section.

CLASSIFICATION of couplings can be made on the basis of rigid or flexible designs.

(A) **Rigid couplings**:

Illustrated by a flange coupling, compression coupling, or tapered-sleeve coupling. This type of coupling is suitable for low speeds, accurately aligned shafts.

(B) **Flexible couplings**:

Illustrated by the Falk flexible coupling, Oldham coupling, gear type of flexible coupling, roller or silent chain coupling, etc.

Flexible couplings are used:

(*a*) To take care of a small amount of unintentional misalignment.

(*b*) To provide for "end float", that is, axial movement of a shaft.

(*c*) To alleviate shock by providing transfer of power through springs or to absorb some of the vibration in the coupling.

Coupling may be classified also as to use, specified by the relation of axes of the connected shafts:

(*1*) Axes of the shafts are collinear.

(2) Axes of the shafts intersect. (A universal joint of the many types available might be used.)

(3) Axes of the shafts are parallel but not collinear. (A coupling of the Oldham type might be used with its central sliding member. This type is to be avoided where possible with heavy loads because of friction due to sliding.)

Since rigid couplings can transmit bending in a shaft, stresses may be induced which can cause fatigue failure. It is therefore desirable to provide for good alignment and location of the coupling where the bending moment is practically zero. Thus rigid couplings, as well as flexible couplings, are usually analyzed for torsion only.

Although standardized couplings may be purchased from manufacturers, the analysis and proportioning of the various parts afford illustration of machine design procedures applied to a single machine element.

SOLVED PROBLEMS

1. A rigid flange coupling has a bore diameter of $\frac{2.000}{2.002}$ in. Four machined bolts on a bolt circle of 5 in. diameter are fitted in reamed holes. If the bolts are made from the same material as the shaft, SAE 1030, with an ultimate tensile strength of 80,000 psi and a yield point in tension of 50,000 psi, determine the necessary size of bolts to have the same capacity as the shaft in torsion. Refer to Fig. 10-1 which shows half a coupling.

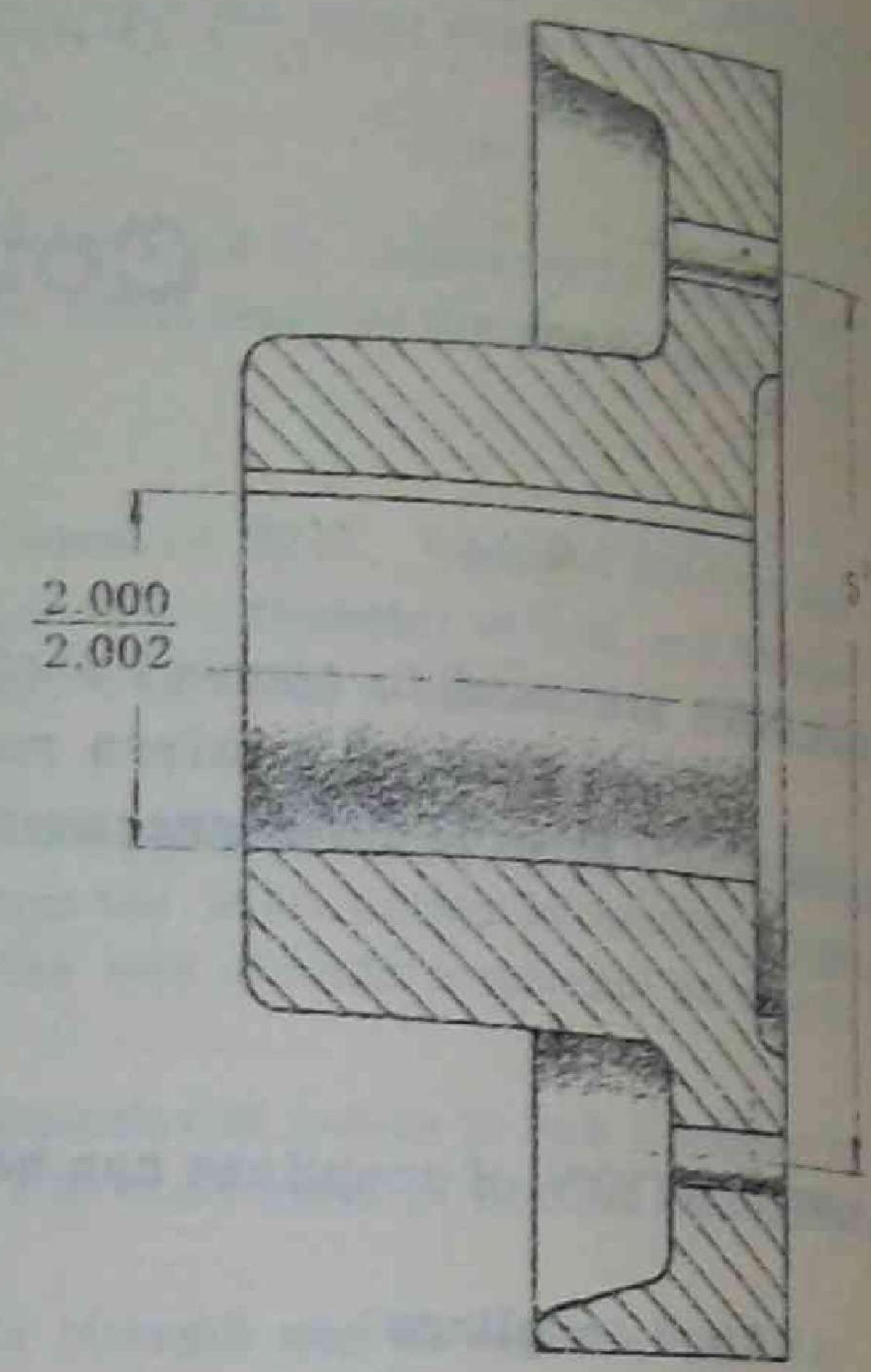


Fig. 10-1

Solution:

(a) The shaft capacity, as determined from the ASME Shafting Code, is found from

$$D^3 = \frac{16}{\pi s_s} M_t K_t$$

the shafting equation for a solid shaft in torsion only. Then

$$(2)^3 = \frac{16}{\pi(14,400)(0.75)} M_t K_t \quad \text{or} \quad M_t K_t = 17,000 \text{ in-lb}$$

where s_s is the smaller of $(0.18)s_u = 0.18(80,000) = 14,400$ psi

and $(0.3)s_{yp} = 0.3(50,000) = 15,000$ psi

and the allowance for the keyway effect is 0.75.

(b) The coupling can be designed for shock and fatigue K_t, equal to 1, or $(M_t K_t)$ can be left as a product and carried through the analysis. The same final result is obtained.

(c) The analysis of the bolts can be made in any one of several different ways:

(1) Assume that the bolts are just finger-tight, and the load is transferred from one half of the coupling to the other half by a uniform shear stress in the shank of the bolt.

(2) Assume that the bolts are just finger-tight, and the load is transferred from one half of the coupling to the other half with a maximum shear stress in the shank of the bolt equal to 4/3 times the average shear stress.

(3) Assume that the bolts are tightened sufficiently so that power is transmitted from one half of the coupling to the other by means of friction.

(4) Assume that the bolts are tightened and that part of the power is transmitted by means of friction and the rest of the power is transmitted by shear in the bolts.

In (1) and (2), it is usual practice to assume that all the bolts share the load proportionally for finished bolts in drilled and reamed holes. (If the bolts are set in clearance holes, it is also usual practice to assume that half the bolts are effective.)

(d) Using arbitrarily (1) above, which gives the most conservative design,

$$M_t K_t = s_s(\tfrac{1}{4}\pi d^2)(\tfrac{1}{2} D_{BC})(n) \quad \text{or} \quad 17,000 = 14,400(\tfrac{1}{4}\pi d^2)(\tfrac{1}{2}\times 5)(4), \quad \text{and} \quad d = 0.387 \text{ in.}$$

where s_s = allowable shear stress, psi

d = diameter of bolt, inches, (shank diameter)

D_{BC} = diameter of bolt circle, inches

n = total number of bolts for drilled and reamed holes.

(Note that s_s for the bolt is the same as obtained from the ASME Shafting Code.) Hence use a 3/8 in. bolt or a 7/16 in. bolt.

(e) Using (2) above, $M_t K_t = \tfrac{3}{4} s_s(\tfrac{1}{4}\pi d^2)(\tfrac{1}{2} D_{BC})(n)$, from which $d = 0.447$ in.; a 7/16 in. bolt may be used.

(f) The next problem will illustrate a solution by (3) above.