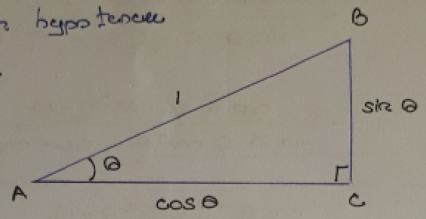
E050, Lesson 5, Trigonometric Identities

TRIGONOMETRIC IDENTITIES

The Pythagorean identities

Triangle ABC with hypotenuse of unit length.



With $\angle BAC = \theta$, $AC = \cos\theta$, $BC = \sin\theta$

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

Dividing this equation respectively by $\cos^2\theta$ and by $\sin^2\theta$,

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad \text{and} \quad \frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta} \quad \text{and} \quad \boxed{\cot^2\theta + 1 = \operatorname{cosec}^2\theta}$$

Example:

Prove the following identities:

(1) $(1-\sin\theta)(1+\sin\theta) = \cos^2\theta$

(2) $$\frac{2\cos^3\theta - \cos\theta}{\sin\theta\cos^2\theta - \sin^3\theta} = \cot\theta$$

Solution: (1)

$$\begin{aligned} LHS &= (1-\sin\theta)(1+\sin\theta) \\ &= 1 - \sin^2\theta \quad \text{(difference of two squares)} \\ &= \cos^2\theta \quad \text{(Pythagorean identity)} \\ &= RHS \end{aligned}$$

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Solution: (2) LHS (Left hand side)

$$= \frac{2\cos^3\theta - \cos\theta}{\sin\theta\cos^2\theta - \sin^3\theta}$$

$$= \frac{\cos\theta\,(2\cos^2\theta - 1)}{\sin\theta\,(\cos^2\theta - \sin^2\theta)}$$

$$= \frac{\cos\theta\,(2\cos^2\theta - 1)}{\sin\theta\,(\cos^2\theta - (1 - \cos^2\theta))}$$

$$= \frac{\cos\theta\,(2\cos^2\theta - 1)}{\sin\theta\,(2\cos^2\theta - 1)}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta$$

$$= RHS$$

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Exercise (1)

(1) Prove that

$$\frac{1}{\sec\theta + \tan\theta} = \frac{\cos\theta}{1 + \sin\theta}$$

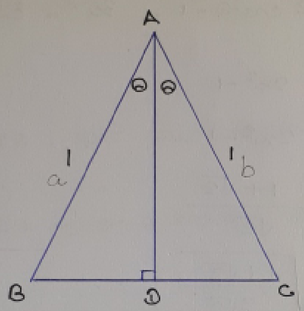
Double angle formulas

The sine and cosine functions are not linear. For example, $\cos(A+B) \neq \cos A + \cos B$. The correct formula for $\cos(A+B)$ is given in the next subsection.

In special case, $A = B = \theta$

simple formula, $\boxed{\cos(2\theta) = \cos 2\theta}$

Double Angle Formula



In triangle ABC, AB = AC = length 1

apex angle = 2θ

Assume, $0° < \theta < 90°$

Let AD be the perpendicular bisector of the interval BC.

By using basic properties of an isosceles triangle, AD bisects $\angle BAC$ and so $BD = DC = \sin\theta$. Applying the cosine rule to the triangle ABC,

$$\cos 2\theta = \frac{\overset{a^2}{1^2} + \overset{b^2}{1^2} - 4\sin^2\theta}{\underset{2ab}{2 \times 1 \times 1}}$$

$$= \frac{2 - 4\sin^2\theta}{2}$$

$$= 1 - 2\sin^2\theta$$

Replacing 1 by $\cos^2\theta + \sin^2\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$ (or)

$\cos 2\theta = 1 - 2\sin^2\theta$

Therefore, $0^\circ < \theta < 90^\circ$, general expression $\cos(A+B)$.

Example: Find $\cos 22\frac{1}{2}^\circ$ in surd form.

Solution: Putting $\theta = 22\frac{1}{2}^\circ$ into the double angle formula

$\cos 2\theta = 2\cos^2\theta - 1$, $x = \cos 22\frac{1}{2}^\circ$,

$$\cos 45^\circ = 2x^2 - 1$$

$$\frac{1}{\sqrt{2}} = 2x^2 - 1$$

$$2x^2 = \frac{1+\sqrt{2}}{\sqrt{2}}$$

$$x = \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}}$$

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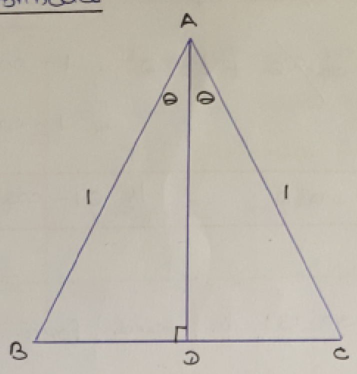
Double Angle Formula



In $\triangle ABC$, on the other hand, the $\boxed{\text{area}}$ $\frac{1}{2} AB \cdot AC \sin(\angle BAC)$

$= \frac{1}{2} \sin 2\theta$

. Since $AD = \cos\theta$, split the triangle into two right-angled triangles

The $\boxed{\text{area}}$ $2 \times \frac{1}{2} \sin\theta \cos\theta = \sin\theta \cos\theta$

Equating these two expressions for the area,

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta} \quad (0^\circ < \theta < 90^\circ)$$

From the Pythagorean identity and the double angle formula for cosine.

$$\cos^2\theta + \sin^2\theta = 1 \quad \text{— eq (1)}$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta \quad \text{— eq (2)}$$

Adding equation (1) & (2),

$$\cos^2\theta + \cancel{\sin^2\theta} + \cos^2\theta - \cancel{\sin^2\theta} = \cos 2\theta + 1$$

$$2\cos^2\theta = \cos 2\theta + 1$$

Dividing by 2, $\boxed{\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)}$ ←

By subtracting these equations,

$$\cancel{\cos^2\theta} + \sin^2\theta - \cancel{\cos^2\theta} + \sin^2\theta = 1 - \cos 2\theta$$

$$2\sin^2\theta = 1 - \cos 2\theta$$

Dividing by 2, $\boxed{\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)}$

Exercise: (1) Find $\sin 15°$ in surd form.

Exercise: (2) Use the double angle formulas for sine and cosine to show that,

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}, \text{ for } \tan\theta \neq \pm 1$$

Summary of double angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}, \text{ for } \tan\theta \neq \pm 1$$

Trigonometric functions of compound angles

sin (A + B)
sin (A − B)
cos (A + B)
cos (A − B)

A compound angle is an algebraic sum of two or more angles.

The formula for trigonometric ratios of compound angles are as follows.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

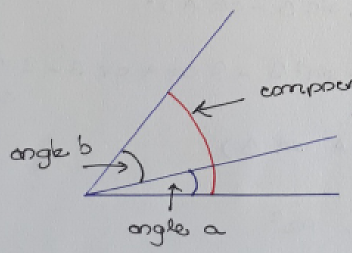
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

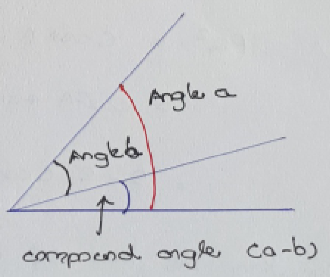
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

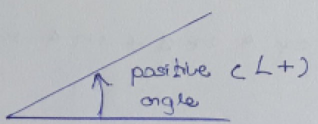
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

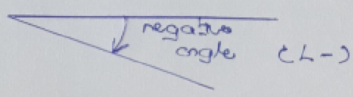
$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$\boxed{\tan(A+B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

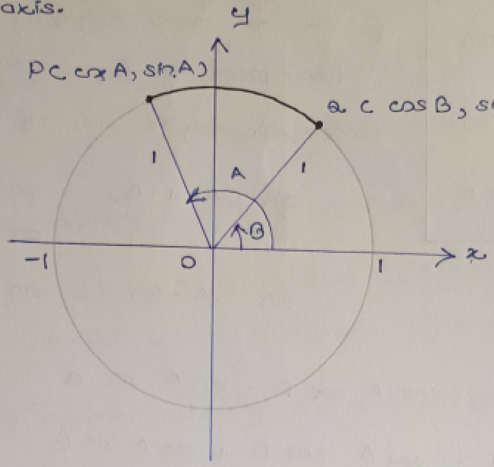








Consider two points $P(\cos A, \sin A)$ and $Q(\cos B, \sin B)$ on the unit circle, taking angle A and B respectively with the positive x-axis.



Apply cosine rule in triangle OPQ, notice that $\angle POQ = A - B$

$$\therefore \cos(\angle POQ) = \cos(A-B)$$

$$PQ^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos(A-B)$$

$$= 2 - \underline{2\cos(A-B)}$$

On the other hand, using the square of distance formula from coordinate geography.

$$PQ^2 = (\cos B - \cos A)^2 + (\sin B - \sin A)^2$$

$$= \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos A \cos B - 2\sin A \sin B$$

$$= 2 - 2(\cos A \cos B + \sin A \sin B)$$

Equating the two ~~diff~~ expressions, for PQ^2

$$2 - 2\cos(A-B) = 2 - 2(\cos A \cos B + \sin A \sin B)$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

If the cosine function is an even function,

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$ (sine function is an odd function)

Therefore, $\cos(A+B) = \cos A \cos(-B) + \sin A \sin(-B)$

$$= \cos A \cos B - \sin A \sin B$$

By using the identity $\sin\theta = \cos(90^\circ - \theta)$,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

standard Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$

Exercise

1. Use the identity $\sin\theta = \cos(90^\circ - \theta)$ to drive the sine expansions.

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