E050, Lesson 9, Probabilities

Probabilities

* Basic Probability Formulas

  - Probability Range $\boxed{0 \leq P(A) \leq 1}$

* Rule of Complementary Events

  $\boxed{P(A^c) + P(A) = 1}$

* Rule of Addition

  $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

* Disjoint Events

  $\boxed{P(A \cap B) = 0}$

* Conditional Probability

  $\boxed{P(A \mid B) = P(A \cap B) / P(B)}$

$\cup \Rightarrow$ union

$\cap \Rightarrow$ intersect ~~or subtract~~

$\phi \Rightarrow$ space

$\cdot \Rightarrow$ product (cross)

$/ \Rightarrow$ subtract

* Bayes Formula

  cross (product)

  $\boxed{P(A \mid B) = P(B \mid A) \cdot P(A) / P(B)}$

* Independent Events

  product (x)

  $\boxed{P(A \cap B) = P(A) \cdot P(B)}$

* Cumulative Distribution Function

  $\boxed{F_X(x) = P(X \leq x)}$

Q. ① Describe the sample space and all 16 events for a trial in which two coins are thrown and each shows either a head or a trial.

Solution: Sample space, $S = \{hh, ht, th, tt\}$

This is 4 elements.

So, $2^4 = 16$ subsets

Namely, ϕ, hh, ht, th, tt, {hh, ht}, {hh, th}, {hh, tt}, ~~{ht, hh}~~, {ht, th}, {ht, tt}, ~~{th, hh}~~, ~~{th, ht}~~, {th, tt}, ~~{tt, hh}~~, ~~{tt, ht}~~, ~~{tt, th}~~, {hh, ht, th}, {hh, ht, tt}, {hh, th, tt}, {ht, th, tt}, {hh, ht, th, tt}

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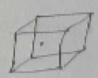
Question ②. A fair coin is tossed, and a fair die is thrown. Write down sample space for



(a) the toss of the coin

(b) the throw of the die

(c) the combination of these experiments.

Let $\boxed{A}$ be the event that a head is tossed, and $\boxed{B}$ be the event that an odd number is thrown. Directly from the sample space, calculate $P(A \cap B)$ and $P(A \cup B)$.

Solution: (a) {Head, Tail}

(b) {1, 2, 3, 4, 5, 6}

(c) {(1 ∩ Head), (1 ∩ Trail), . . . , (6 ∩ Head), (6 ∩ Trail)}

Clearly $P(A) = \frac{1}{2} = P(B)$, $P(A \cap B) = P(A) \cdot P(B) = \frac{1}{4}$

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Question (4). A bag contains fifteen balls distinguishable only by their colours; ten are blue and five are red. I reach into the bag with both hands and pull out two balls (one with each hand) and record their colours.

(a) What is the random phenomenon?

(b) What is the sample space?

(c) Express the event that the ball in my left hand is red as a subset of the sample space.

Solutions:

(a) The random phenomenon is the colours of the two balls.

(b) The sample space is the set of all possible colours for the two balls,

$$\{(B,B),\ (B,R),\ (R,B),\ (R,R)\}$$

(c) The event is the subset $\{(R,B),\ (R,R)\}$ #

Question (3). M & M sweets are of varying colours and the different colours occur in different proportions. The table below gives the probability that a randomly chosen M & M has each colour, but the value for tan candies is missing.

Colour	Brown	Red	Yellow	Green	Orange	Tan
Probability	0.3	0.2	0.2	0.1	0.1	?

(a) What value must the missing probability be?

(b) You draw an M & M at random from a pocket. What is the probability of each of the following events?

(i) You get a brown one, or a red one.

(ii) You don't get a yellow one.

(iii) You don't get either an orange one or a tan one.

(iv) You get one that is brown or red or yellow or green or orange or tan.

Solution: (a) The probability must sum to 1.0.

Therefore, $1 - 0.3 - 0.2 - 0.2 - 0.1 - 0.1 = 1 - 0.9 = .1$ ⇐

(b) (i) $0.3 + 0.2 = 0.5$, (it can't be brown and red simultaneously)

(ii) $1 - P(\text{yellow}) = 1 - 0.2 = 0.8$. ⇐

(iii) $1 - P(\text{orange or tan}) = 1 - P(\text{orange}) - P(\text{tan})$

$= 1 - 0.1 - 0.1$

$= 0.8$ ⇐

(iv) must happen, probability is 1.0. ⇐

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Question ⑤. Not all dice are fair, In order to describe an unfair die properly, we must specify the probability for each of the six possible outcomes. The following table gives answers for each of 4 different dice.

	Probabilities			
Outcome	Die 1	Die 2	Die 3	Die 4
1	$1/3$	$1/6$	$1/7$	$1/3$
2	0	$1/6$	$1/7$	$1/3$
3	$1/6$	$1/6$	$1/7$	$-1/6$
4	0	$1/6$	$1/7$	$-1/6$
5	$1/6$	$1/6$	$1/7$	$1/3$
6	$1/3$	$1/7$	$2/7$	$1/3$

Which of the four dice have validity specified probabilities and which do not. In the case of an invalidity described die, explain why the probabilities are valid.

Solution:

(a) Die 1 is valid.

(b) Die 2 is invalid, The probability do not sum to 1.

(c) Die 3 is valid.

(d) Die 4 is invalid. Two of the probabilities are negative.

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Exercises (1)

1. Toss a coin three times, what is the probability of at least two heads?

2. One card is drawn from a standard pack of 52 playing cards. What is the probability of (a) picking a red card (b) picking a king (c) picking a diamond?

3. What is the probability of throwing a total score of 6 with two dice?

Addition of Probabilities

vowel = A or E or I or O or U

number of vowels → $n(\text{vowels}) = n(A) + n(E) + n(I) + n(O) + n(U)$

$P(\text{Vowel}) = P(A) + P(E) + P(I) + P(O) + P(U)$

Example: What is the probability of drawing an ace or a spade from a well-shuffled pack of cards?

solution: 4 aces, $P(\text{ace}) = \frac{4}{52}$

There are 13 spades ; $P(\text{spade}) = \frac{13}{52}$

The probability of an ace or a spade $= \frac{4+13}{52} = \frac{17}{52}$

This is incorrect. The problem is that the ace of spades has been counted twice, once as an ace and once as a spade.

The correct answer is $\frac{4+13-1}{52} = \frac{16}{52} = \frac{4}{13}$ #

Exercise (7)

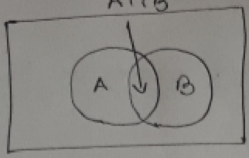
1. A bag contains letter. Scrabble pieces has the following letter distribution.

A	B	C	D	E	F	G	H	I	J	K	L	M
9	2	2	4	12	2	3	2	9	1	1	4	2
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
6	8	2	1	6	4	6	4	2	2	1	2	1

The first letter is chosen at random from the bag. Find the probability that it is: (i) an E ; (ii) in the first half of the alphabet (iii) in the second half of the alphabet (iv) a vowel (v) a consonant (vi) the only one of its kind.

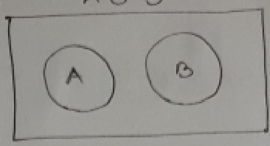
↓ intersect

$A \cap B$



↓ union

$A \cup B$



$A \cup B = \phi \quad P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Exclusive events : $P(A \cup B) = P(A) + P(B)$

P7

Exercise 3 : 100 cars are entered for a road-worthiness test which is in two parts, mechanical and electrical. A car passes only if it passes both parts. Half the cars fails the electrical test and 62 pass the mechanical. 15 pass the electrical but fail the mechanical test. Find the probability that a car chosen at random.

(i) passes overall (ii) fails on one test only

(iii) given that it has failed, failed the mechanical test only.

Mean, Variance and Standard Deviation

Arithmetic mean: $\mu = \frac{\sum_{i=1}^{N} x_i}{N}$,

Variance: $\sigma^2 = \frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N}$

Standard Deviation: $\sigma = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N}}$

Example: Imagine a pack of cards with all the jokers and picture cards removed. We are only concerned with the numerical value of the cards. We have four each of all the numbers from one to ten so the pack contains 40 cards.

Solution:

Arithmetic mean: $\mu = \frac{\sum_{i=1}^{N} x_i}{N} = \frac{220}{40} = 5.5$

Variance: $\sigma^2 = \frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N} = \frac{330}{40} = 8.25$

Standard Deviation: $\sigma = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N}}$

$= \sqrt{\frac{330}{40}}$

$= 2.8723$

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(P2)

Exercise (4): For each of the cases below, either find a set of five numbers that satisfies the required condition or explain why it cannot be achieved.

(i) Four of the five numbers are below the mean.

(ii) The mean is less than the standard deviation.

(iii) The mean is greater than the standard deviation.

(iv) The standard deviation is below the variance.
